Generalized nonlinear heat and Navier-Stokes equations in supercritical function spaces

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FÜR CORA

Zusammenfassung

Die vorliegende Arbeit wurde vor allem durch die Ergebnisse Triebels in [37] motiviert. Dort werden unter anderem die klassischen Navier-Stokes Gleichungen

$$\begin{aligned} \partial_t \boldsymbol{u} - \Delta_x \boldsymbol{u} + (\boldsymbol{u}, \nabla)\boldsymbol{u} + \nabla P &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0 \quad \text{in } \mathbb{R}^n \end{aligned} \tag{0.0.1}$$

vor dem Hintergrund superkritischer Funktionenräume vom Besov- und Triebel-Lizorkin-Typ untersucht. Die Gleichungen (0.0.1) beschreiben die Bewegung einer Flüssigkeit im $\mathbb{R}^n$, wobei $n = 2$ oder $n = 3$ die physikalisch relevanten Fälle darstellen, ausgehend von einem gegebenen Anfangsgeschwindigkeitsfeld $\boldsymbol{u}_0$. Gesucht sind das Geschwindigkeitsfeld $\boldsymbol{u}(x,t) = (u_1(x,t), \cdots, u_n(x,t)) \in \mathbb{R}^n$ und der Druck $P(x,t) \in \mathbb{R}$ zur Zeit $t > 0$ am Ort $x \in \mathbb{R}^n$. Unter der Bedingung $\operatorname{div} \boldsymbol{u} = 0$ gilt $(\boldsymbol{u}, \nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})$, wobei $\otimes$ das Tensorprodukt bezeichnet. Mit Hilfe des Lerayprojektors $\mathbb{P}$ wird unter Ausnutzung dieser Beziehung (0.0.1) wie folgt umgeformt. Wir betrachten

$$\begin{aligned} \partial_t \boldsymbol{u} - \Delta_x \boldsymbol{u} + \mathbb{P} \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{0.0.2}$$

Diese Umformulierung hat den Vorteil, dass die vektorwertigen Navier-Stokes Gleichungen auf den eindimensionalen Fall, genauer gesagt eine klassische Wärmeleitungsgleichung mit spezieller Nichtlinearität reduziert werden können. Den zweiten Impuls für unsere Untersuchungen gab die Arbeit Miaos, Yuans und Zhangs [28]. Hier wird der Diffusionsoperator $-\Delta_x$ in (0.0.1) durch eine Potenz $(-\Delta_x)^\alpha$ ersetzt, wobei $\alpha > 0$ reelle Werte annimmt. Diese Verallgemeinerung führt bei der Wahl einer geeigneten Nichtlinearität zum Beispiel für $1/2 < \alpha \leq 1$ auf eine quasigeostrophische dissipative Gleichung oder für $\alpha > 0$ auf eine verallgemeinerte Konvektions-Diffusions-Gleichung. Hier ist insbesondere der Fall $\alpha = 2$ interessant, wenn die Nichtlinearität in (0.0.2) durch den Term

$$\mathbb{P}(\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x |u|^2) \tag{0.0.3}$$

ersetzt wird. Dieses vereinfachte Modell vereint die Beschreibung von inkompressiblen Flüssigkeiten mit der eigenständigen Bewegung darin lebender Mikroorganismen, z.B. *Bacillus subtilis*. Solche Bewegungen werden unter anderem durch chemische Substanzen ausgelöst, die von den Organismen entweder selbst produziert oder aufgenommen werden. Man spricht dabei von Chemotaxis. Erste Ergebnisse dazu findet man z.B.

in [11], [12] und [46]. Eine umfassende mathematische Beschreibung fehlt bislang. Wünschenswert wäre eine qualitative Methode, die die Abbildungseigenschaften des Operators $\partial_t + (-\Delta_x)^\alpha$ für beliebige Werte $\alpha > 0$ charakterisiert. Einen ersten Ausgangspunkt dazu kann die vorliegende Arbeit bilden.
Wir starten mit der verallgemeinerten nichtlinearen Wärmeleitungsgleichung

$$\begin{aligned} \partial_t u + (-\Delta_x)^\alpha u - Du^2 &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \mathbb{R}^n, \end{aligned} \tag{0.0.4}$$

wobei $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ und $Du^2 = \sum\limits_{j=1}^{n} \partial_j u^2$. Der Fall $\alpha = 1$ entspricht damit einer klassischen nichtlinearen Wärmeleitungsgleichung. Wir benutzen (0.0.4) als skalaren Modellfall für die mittels Lerayprojektor $\mathbb{P}$ umgeformten verallgemeinerten Navier-Stokes Gleichungen

$$\begin{aligned} \partial_t \boldsymbol{u} + (-\Delta_x)^\alpha \boldsymbol{u} + (\boldsymbol{u}, \nabla)\boldsymbol{u} + \nabla P &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0 && \text{in } \mathbb{R}^n, \end{aligned} \tag{0.0.5}$$

d.h. wir interessieren uns für Lösungen folgenden Gleichungssystems

$$\begin{aligned} \partial_t \boldsymbol{u} + (-\Delta_x)^\alpha \boldsymbol{u} + \mathbb{P} \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0 && \text{in } \mathbb{R}^n \end{aligned} \tag{0.0.6}$$

mit Parametern n, α und T wie oben angegeben. Unter gewissen Voraussetzung an die betrachteten Funktionenräume können alle Ergebnisse für (0.0.4) auf (0.0.6) übertragen werden.
Äquivalent zur Lösung von (0.0.4) ist die Lösung des folgenden Fixpunktproblems für den Operator T_{u_0}, definiert als

$$T_{u_0} u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x,\tau) \mathrm{d}\tau, \quad x \in \mathbb{R}^n, \ 0 < t < T, \tag{0.0.7}$$

auf einem gewichteten Lebesgueraum $L_v((0,T),\, b,\, X)$, siehe (0.0.9) und (0.0.10). Hierbei ist

$$W_t^\alpha \omega(x) := \left[\frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}} \right)^\vee * \omega \right](x), \quad t > 0,\, x \in \mathbb{R}^n \tag{0.0.8}$$

und $\omega \in S'(\mathbb{R}^n)$. Dabei fragen wir nach Lösungen, die bezüglich der Raumvariablen in Funktionenräumen $A_{p,q}^s(\mathbb{R}^n)$ mit $A \in \{B, F\}$ vom Besov- bzw. Triebel-Lizorkin Typ liegen. Aufgrund der speziellen Struktur der Nichtlinearität müssen die Parameter s, p und q so gewählt sein, dass die punktweise Multiplikation von Elementen dieser Räume definiert ist. Wir betrachten zunächst Multiplikationsalgebren, d.h. das Produkt zweier Elemente dieser Räume gehört wieder zum selben Raum. Für $s > n/p$ und in einigen

Grenzfällen $s = n/p > 0$ ist dies der Fall. Danach schwächen wir diese Bedingung ab zu $n/p - 1 < s < n/p$, $s > 0$. Dann ist die punktweise Multiplikation immer noch definiert, allerdings gehört das Produkt nicht mehr zum Ausgangsraum. Weiterhin untersuchen wir den Grenzfall $s = n/p > 0$ in voller Allgemeinheit und den Spezialfall $F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ für $2 \le n < p < \infty$. Die Abhängigkeit von der Zeit t wird mit Hilfe gewichteter Lebesgueräume $L_v((0,T),\, b,\, X)$ beschrieben. Diese Räume sind definiert als die Menge aller Funktionen $u : (0,T) \to X$, sodass

$$\int_0^T t^{bv} \|u(\cdot,t)|X\|^v \mathrm{d}t < \infty, \quad \text{falls } v < \infty, \tag{0.0.9}$$

beziehungsweise

$$\sup_{0<t<T} t^b \|u(\cdot,t)|X\|, \quad \text{falls } v = \infty, \tag{0.0.10}$$

gilt, wobei $1 \le v \le \infty$, $b \in \mathbb{R}$, $0 < T < \infty$ und $X = A^s_{p,q}(\mathbb{R}^n)$ ein Banachraum ist.
Wir konzentrieren uns zunächst auf die Lösung des homogenen Problems

$$\begin{array}{rcll} \partial_t u + (-\Delta_x)^\alpha u &=& 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) &=& u_0(x) & \text{in } \mathbb{R}^n \end{array} \tag{0.0.11}$$

und untersuchen Abbildungseigenschaften von (0.0.8) in $A^s_{p,q}(\mathbb{R}^n)$. Dabei nutzen wir aus, dass Elemente dieser Räume mittels geeigneter Bausteine, in unserem Falle Wavelets und Moleküle, zerlegt werden können. Konkret werden dabei Eigenschaften von Funktionenräumen auf Eigenschaften von Folgenräumen reduziert. Als erstes Hauptresultat beweisen wir die Abschätzung

$$t^{d/2\alpha} \|W^\alpha_t \omega | A^{s+d}_{p,q}(\mathbb{R}^n)\| \le c \|\omega | A^s_{p,q}(\mathbb{R}^n)\|, \quad 0 < t \le 1 \tag{0.0.12}$$

für $1 \le p, q \le \infty$ ($p < \infty$ im Falle der F- Räume), $s \in \mathbb{R}$, $d \ge 0$ und $\alpha \in \mathbb{N}$.
Mit Hilfe dieser Abschätzung lösen wir das Fixpunktproblem (0.0.7) unter Anwendung des Banachschen Fixpunktsatzes. Lösungen einer partiellen Differentialgleichung, die aus einem Fixpunktproblem hervorgehen, nennt man *milde* Lösungen. Ist diese Lösung zusätzlich stetig bezüglich der Zeit bis in den Nullpunkt hinein, gemessen in der Norm des Raumes, zu welchem die Anfangsbedingungen gehören, spricht man von *starken* Lösungen. Wir zeigen, dass die Probleme (0.0.4) - (0.0.6) unter der Wahl geeigneter Anfangsbedingungen eindeutig bestimmte starke Lösungen in gewichteten vektorwertigen Lebesgueräumen besitzen. Insbesondere sind diese Lösungen bezüglich der Zeit- und Raumvariablen beliebig oft differenzierbar. Weiterhin zeigen wir, dass die Probleme (0.0.4) - (0.0.6) unter diesen Bedingungen korrekt gestellt sind im Sinne von [40, Kap. 4.6].
Die Arbeit ist wie folgt aufgebaut. Im Kapitel 1 legen wir die Notation fest und stellen alle erforderlichen technischen Hilfsmittel zur Verfügung. Insbesondere gehen wir dabei

auf die Definitionen von Wavelets und Molekülen ein und wiederholen die wesentlichen Resultate zur Charakterisierung von Besov- und Triebel-Lizorkin Räumen mit Hilfe solcher Bausteine.
Im Kapitel 2 bereiten wir die Konstruktion der Lösungen von (0.0.4) - (0.0.6) als Lösung der Fixpunktgleichung $T_{u_0}u = u$ mit T_{u_0} wie in (0.0.7) definiert, vor. Wir zeigen, dass der Faltungskern von (0.0.8),

$$G^\alpha(x,t) = \begin{cases} \frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee (x), & t > 0,\ x \in \mathbb{R}^n, \\ 0, & t \leq 0,\ x \in \mathbb{R}^n, \end{cases} \tag{0.0.13}$$

eine Fundamentallösung der verallgemeinerten Wärmeleitungsgleichung ist, d.h. es gilt

$$(\partial_t + (-\Delta_x)^\alpha)G^\alpha = \delta,$$

wobei δ die δ- Distribution bezeichnet, und untersuchen das Cauchyproblem für (0.0.4) im Distributionensinn. Insbesondere verifizieren wir, dass unsere Lösungsräume ausschließlich reguläre Distributionen enthalten und somit die Lösungsdarstellung als Faltung mit dem Kern G^α gerechtfertigt ist. Abschließend stellen wir die fourieranalytischen Methoden zur Behandlung der verallgemeinerten Navier-Stokes Gleichungen und ihrer Umformulierung, (0.0.5) und (0.0.6), bereit.
Im Kapitel 3 untersuchen wir zunächst die Abbildungseigenschaften von W_t^α und beweisen die Abschätzung (0.0.12) in geeigneten Funktionenräumen $A^s_{p,q}(\mathbb{R}^n)$. Dabei verwenden wir eine Zerlegung von ω mit Hilfe von Daubechies Wavelets. Bei der Faltung dieser Wavelets mit G_t^α ($= G^\alpha$ für festes t) geht der kompakte Träger verloren. Wir zeigen, dass mit Hilfe dieser neuen Bausteine, sogenannter α-kalorischer Wavelets, eine molekulare Zerlegung der Räume $A^s_{p,q}(\mathbb{R}^n)$ gelingt. Als Hauptresultat beweisen wir die Existenz eindeutig bestimmter starker Lösungen von (0.0.4) unter der Bedingung, dass die zugrunde liegenden Räume $A^s_{p,q}(\mathbb{R}^n)$ Multiplikationsalgebren sind und untersuchen ihre Stabilität, d.h. ihre Veränderung in Abhängigkeit der Anfangsdaten.
Im Kapitel 4 wenden wir uns der Frage zu, inwieweit die Glattheitseigenschaften der Räume $A^s_{p,q}(\mathbb{R}^n)$ abgeschwächt werden können und nutzen dafür Hölderungleichungen und Einbettungsresultate aus Kapitel 1.4. Dafür klären wir zunächst, wann ein Funktionenraum in Abhängigkeit der betrachteten Differentialgleichung als kritisch, sub- oder superkritisch bezeichnet wird. Weiterhin untersuchen wir die Grenzfälle $s = n/p$ und den Fall $s = 0$ im Rahmen superkritischer Lebesgueräume $L_p(\mathbb{R}^n)$, $2 \leq n < p < \infty$.
Im Kapitel 5 wenden wir die Resultate aus den Kapiteln 3 und 4 auf die umformulierte Version der verallgemeinerten Navier-Stokes Gleichungen, also auf (0.0.6) an. Dabei nutzen wir die Abbildungseigenschaften des Lerayprojektors und insbesondere der Riesztransformation in den Räumen $A^s_{p,q}(\mathbb{R}^n)$ mit $1 < p < \infty$, $1 \leq q \leq \infty$ aus. Weiterhin klären wir die Bedingungen, unter denen (0.0.5) und (0.0.6) äquivalent sind.

Introduction

The present thesis was essentially motivated by the results by Triebel presented in [37] where he considers the classical Navier-Stokes equations

$$\begin{array}{rcll} \partial_t \boldsymbol{u} - \Delta_x \boldsymbol{u} + (\boldsymbol{u}, \nabla)\boldsymbol{u} + \nabla P & = & 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \boldsymbol{u} & = & 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) & = & \boldsymbol{u}_0 & \text{in } \mathbb{R}^n \end{array} \tag{0.0.14}$$

in supercritical function spaces of Besov and Triebel-Lizorkin type. The equations (0.0.14) describe the motion of a fluid in $\mathbb{R}^n$, whereas $n = 2$ or $n = 3$ are the physically relevant cases. One looks for a velocity vector $\boldsymbol{u}(x,t) = (u_1(x,t), \cdots, u_n(x,t)) \in \mathbb{R}^n$ and a pressure $P(x,t) \in \mathbb{R}$ under a given inital velocity field. From the condition $\operatorname{div} \boldsymbol{u} = 0$ it follows $(\boldsymbol{u}, \nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})$ where $\otimes$ denotes the usual tensor product. Using this relation one transforms (0.0.14) into

$$\begin{array}{rcll} \partial_t \boldsymbol{u} - \Delta_x \boldsymbol{u} + \mathbb{P} \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) & = & 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) & = & \boldsymbol{u}_0 & \text{in } \mathbb{R}^n. \end{array} \tag{0.0.15}$$

This reformulation has the advantage that the system of Navier-Stokes equations can be reduced to a scalar nonlinear heat equation. Another essential impetus for our considerations comes from the paper by Miao, Yuan and Zhang [28]. They replace the dissipative operator $-\Delta_x$ in (0.0.14) by a power $(-\Delta_x)^\alpha$ whereas $\alpha > 0$ takes real values. Regarding suitable nonlinearities this generalization leads for example for $1/2 < \alpha \leq 1$ to a so-called quasi-geostrophic dissipative equation or for $\alpha > 0$ to a generalized convection-diffusion equation. Of particular interest is the case $\alpha = 2$ when one replaces the nonlinearity in (0.0.15) by

$$\mathbb{P}(\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x |u|^2). \tag{0.0.16}$$

This simplified model combines the description of incompressible fluids with the self-sustained motion of living matters, for instance *bacillus subtilis*. Among other influences such dynamical phases like formation of flocks are forced by chemicals, either consumed or produced by the organisms themselves. This phenomenon is called chemotaxis. For first results we refer i.a. to [11], [12] und [46]. Up to now a comprehensive description is not available. It would be desirable to have a qualitative method which characterizes the mapping properties of the operator $\partial_t + (-\Delta_x)^\alpha$ for arbitrary values of $\alpha > 0$. The present paper could serve as a first starting point.

To begin with we consider the generalized nonlinear heat equation

$$\begin{aligned} \partial_t u + (-\Delta_x)^\alpha u - Du^2 &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \mathbb{R}^n \end{aligned} \tag{0.0.17}$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $Du^2 = \sum_{j=1}^{n} \partial_j u^2$. Thus, the case $\alpha = 1$ corresponds to a classical nonlinear heat equation. Further, we are interested in solutions of the generalized Navier-Stokes equations

$$\begin{aligned} \partial_t \boldsymbol{u} + (-\Delta_x)^\alpha \, \boldsymbol{u} + (\boldsymbol{u}, \nabla)\boldsymbol{u} + \nabla P &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0 && \text{in } \mathbb{R}^n \end{aligned} \tag{0.0.18}$$

reformulated by means of the Leray projector using (0.0.17) as their scalar model case. More precisely we consider the Cauchy problem

$$\begin{aligned} \partial_t \boldsymbol{u} + (-\Delta_x)^\alpha \, \boldsymbol{u} + \mathbb{P} \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0 && \text{in } \mathbb{R}^n \end{aligned} \tag{0.0.19}$$

with parameters n, α and T as above. Under certain requirements on the underlying function spaces it is possible to transfer all results gained for (0.0.17) to (0.0.19). We consider solutions of (0.0.17) which solve the fixed point problem $T_{u_0} u = u$ whereas the operator T_{u_0} is defined as

$$T_{u_0} u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x,\tau) \mathrm{d}\tau, \quad x \in \mathbb{R}^n, \; 0 < t < T \tag{0.0.20}$$

in some weighted Lebesgue spaces $L_v((0,T), \, b, \, X)$, see (0.0.22) and (0.0.23) below. Here $W_t^\alpha \omega$ is defined as

$$W_t^\alpha \omega(x) := \left[\frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}} \right)^\vee * \omega \right](x), \quad t > 0, \, x \in \mathbb{R}^n \tag{0.0.21}$$

for $\omega \in S'(\mathbb{R}^n)$. In particular we ask for solutions of (0.0.20) which belong to some spaces $A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ of Besov or Triebel-Lizorkin type with respect to the space variable. Because of the structure of the nonlinearity the parameters s, p and q must be chosen such that the pointwise multiplication of two elements of these spaces is defined. We start with the case when $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra. Then the product of two elements belongs again to the same space. This holds true if $s > n/p$ and in some limiting cases $s = n/p > 0$. Then we relax this condition to $n/p - 1 < s < n/p$, $s > 0$. Here the pointwise multiplication is still defined but the product does not belong to the original space. Moreover, we investigate the limiting case $s = n/p > 0$ in full generality and the special case $F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ with $2 \leq n < p < \infty$. The dependence on time will be described in terms of weighted Lebesgue spaces $L_v((0,T),\, b,\, X)$ defined as the collection of all functions $u:(0,T)\to X$ such that

$$\int_0^T t^{bv}\|u(\cdot,t)|X\|^v \mathrm{d}t < \infty, \quad \text{if } v<\infty \tag{0.0.22}$$

and

$$\sup_{0<t<T} t^b\|u(\cdot,t)|X\|, \quad \text{if } v=\infty, \tag{0.0.23}$$

respectively, with $1\le v\le\infty$, $b\in\mathbb{R}$, $0<T<\infty$ holds. In our case we choose the Banach space $X=A^s_{p,q}(\mathbb{R}^n)$.
First we focus on the homogeneous problem

$$\begin{aligned} \partial_t u + (-\Delta_x)^\alpha u &= 0 \qquad \text{in } \mathbb{R}^n\times(0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \mathbb{R}^n \end{aligned} \tag{0.0.24}$$

and consider mapping properties of W^α_t defined in (0.0.21) in $A^s_{p,q}(\mathbb{R}^n)$. To this end we use decomposition methods by means of wavelets and molecules. This reduces properties of function spaces to properties of sequence spaces. As our first main result we prove the estimate

$$t^{d/2\alpha}\|W^\alpha_t\omega|A^{s+d}_{p,q}(\mathbb{R}^n)\| \le c\|\omega|A^s_{p,q}(\mathbb{R}^n)\|, \quad 0<t\le 1 \tag{0.0.25}$$

with $1\le p,q\le\infty$ ($p<\infty$ in case of F- spaces), $s\in\mathbb{R}$, $d\ge 0$ and $\alpha\in\mathbb{N}$.
Applying Banach's contraction principle and using (0.0.25) we solve the fixed point problem (0.0.20). Solutions of a PDE which come out of a fixed point problem are called *mild* solutions. If this solution belongs additionally to the space $C((0,T),A^{s_0}_{p,q}(\mathbb{R}^n))$ for all initial data $u_0\in A^{s_0}_{p,q}(\mathbb{R}^n)$ then the solution is called *strong*. For more explanations we refer to [5] and [23]. We show the existence of unique strong solutions of (0.0.17) - (0.0.19) in weighted vector-valued Lebesgue spaces under the choice of suitable initial data. It turns out that these solutions are C^∞- functions with respect to space and time. Moreover, under these conditions the problems (0.0.17) - (0.0.19) are well-posed in the sense of [40, Kap. 4.6].
The paper is organized as follows. In Chapter 1 we fix notation and provide basic results and tools which will be used later on. In particular we introduce the concept of Daubechies wavelets and molecules (as far as we need them for our considerations). Further we recall the characterization of the spaces $A^s_{p,q}(\mathbb{R}^n)$ via these wavelets and molecules.
In Chapter 2 we prepare the construction of solutions of (0.0.17) - (0.0.19) as solutions of the fixed-point equation related to (0.0.20). In particular we show that the kernel

$$G^\alpha(x,t) = \begin{cases} \frac{1}{(2\pi)^{n/2}}\left(e^{-t|\xi|^{2\alpha}}\right)^\vee(x), & t>0,\ x\in\mathbb{R}^n, \\ 0, & t\le 0,\ x\in\mathbb{R}^n \end{cases} \tag{0.0.26}$$

of (0.0.21) is a fundamental solution of the generalized heat equation, i.e. it holds

$$(\partial_t + (-\Delta_x)^\alpha)G^\alpha = \delta$$

where δ denotes the δ- distribution, and consider Cauchy's problem for (0.0.17) in the distributional sense. It comes out that our solution spaces consist entirely of regular distributions. Hence, the representation as convolution with kernel G^α is justified. Finally we discuss the Fourier-analytical methods which are necessary to deal with the generalized Navier-Stokes equations and its reformulation.
In Chapter 3 we consider at first mapping properties of W_t^α and prove the estimate (0.0.25). To this end we use a decomposition of ω by means of Daubechies wavelets. After the convolution with G_t^α ($= G^\alpha$ for fixed t) the resulting building blocks, which are called α- caloric wavelets, don't have a compact support. We show that after a slight modification they are molecules for appropriate function spaces. As main result we show the existence of unique strong solutions of (0.0.17) under the condition that the underlying function spaces with respect to the space variable are multiplication algebras. Further, we show that the problem is well-posed.
Chapter 4 deals with the same problem under weaker conditions on the smoothness of the solution. In particular we focus on the strip $n/p - 1 < s < n/p$. To begin with we clarify which spaces should be called critical, sub- and supercritical in this context. We investigate the limiting case $s = n/p$ and the interesting case $s = 0$ in the context of supercritical Lebesgue spaces $L_p(\mathbb{R}^n)$, $2 \leq n < p < \infty$.
Finally, in Chapter 5 we apply the results related to the scalar case to the reformulated Navier-Stokes equations. Thereby we use mapping properties of the Leray projector and in particular of the Riesz transform. Further we clarify under which conditions (0.0.18) and (0.0.19) are equivalent.

Contents

1 Function spaces on $\mathbb{R}^n$

1.1 Notation

Let $\mathbb{R}^n$ be Euclidean n- space with $n \in \mathbb{N}$, where $\mathbb{N}$ indicates the collection of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : \ x \in \mathbb{R}^n, \ t > 0\}$$

and $\overline{\mathbb{R}^{n+1}_+}$ its closure. We put $\mathbb{R} = \mathbb{R}^1$. As usual $\mathbb{C}$ denotes the complex plane. Let $\mathbb{Z}$ be the collection of all integers and $\mathbb{Z}^n$, $n \in \mathbb{N}$, the lattice of all points $m = (m_1, \ldots, m_n)$ with $m_j \in \mathbb{Z}$, $j = 1, \ldots, n$. $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, denotes the set of all multi-indices

$$\gamma = (\gamma_1, \ldots, \gamma_n) \text{ with } \gamma_j \in \mathbb{N}_0 \text{ and } |\gamma| = \sum_{j=1}^{n} \gamma_j .$$

For $x \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}_0^n$ we write

$$x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \text{ and } \mathrm{D}^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \ldots \partial x_n^{\gamma_n}} .$$

Further, we abbreviate $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ and $\partial_j^m = \partial^m/\partial x_j^m$, $m \in \mathbb{N}_0$. We write $a_+ = \max(a, 0)$, $a \in \mathbb{R}$ and define for fixed $n \in \mathbb{N}$

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \qquad \sigma_{p,q} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{1.1.1}$$

where $0 < p, q \le \infty$. Any unimportant constant will be denoted by c or C.

1.2 Definitions and basic properties

In this section we introduce the function spaces we are dealing with in the sequel, some of their basic properties as well as some special cases. As usual $S(\mathbb{R}^n)$ denotes the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ and $S'(\mathbb{R}^n)$ its dual, the space of all tempered distributions. Furthermore, $L_p(\mathbb{R}^n)$ with

$0 < p \le \infty$ is the space of all p- integrable complex-valued functions with respect to the Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p}. \tag{1.2.1}$$

If $p = \infty$ we modify (1.2.1) by

$$\|f|L_\infty(\mathbb{R}^n)\| = \underset{x\in\mathbb{R}^n}{\text{ess-sup}}|f(x)|. \tag{1.2.2}$$

The Fourier transform of $\phi \in S(\mathbb{R}^n)$ is defined by

$$\widehat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\phi(x)\mathrm{d}x, \quad \xi \in \mathbb{R}^n. \tag{1.2.3}$$

$\mathcal{F}^{-1}\phi$ and $\phi^\vee$ stand for the inverse Fourier transform, given by the right hand side of (1.2.3) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended in the usual way to $S'(\mathbb{R}^n)$. Let $\phi_0 \in S(\mathbb{R}^n)$ with

$$\phi_0(x) = 1 \text{ if } |x| \le 1 \text{ and } \phi_0(x) = 0 \text{ if } |x| \ge 3/2. \tag{1.2.4}$$

We define the sequence

$$\phi_k(x) = \phi_0(2^{-k}x) - \phi_0(2^{-k+1}x) \text{ for } x \in \mathbb{R}^n,\ k \in \mathbb{N}. \tag{1.2.5}$$

Then

$$\sum_{k=0}^{\infty} \phi_k(x) = 1 \text{ for all } x \in \mathbb{R}^n \tag{1.2.6}$$

and $\{\phi_k\}_{k=0}^\infty$ is called a smooth dyadic resolution of unity. Because of the Paley-Wiener-Schwartz theorem $\mathcal{F}^{-1}[\phi_k\mathcal{F}f]$ are entire analytic functions for all $f \in S'(\mathbb{R}^n)$ and hence make sense pointwise.

Definition 1.2.1. Let $\phi = \{\phi_k\}_{k=0}^\infty$ be the above dyadic resolution of unity.

(i) For $0 < p, q \le \infty$, $s \in \mathbb{R}$ we define the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ as the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|_\phi = \left(\sum_{k=0}^{\infty} 2^{ksq}\left\|\mathcal{F}^{-1}\phi_k\mathcal{F}f\middle|L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} \tag{1.2.7}$$

is finite. If $q = \infty$ we replace (1.2.7) by

$$\|f|B^s_{p,\infty}(\mathbb{R}^n)\|_\phi = \sup_{k\in\mathbb{N}_0} 2^{ks}\left\|\mathcal{F}^{-1}\phi_k\mathcal{F}f\middle|L_p(\mathbb{R}^n)\right\|. \tag{1.2.8}$$

(ii) For $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$ the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are defined as the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|F^s_{p,q}(\mathbb{R}^n)\|_\phi = \Big\| \Big(\sum_{k=0}^{\infty} 2^{ksq} |\mathcal{F}^{-1}\phi_k \mathcal{F} f(\cdot)\Big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\| \tag{1.2.9}$$

is finite. If $q = \infty$ we replace (1.2.9) by

$$\|f|F^s_{p,\infty}(\mathbb{R}^n)\|_\phi = \Big\| \sup_{k \in \mathbb{N}_0} 2^{ks} \, |\mathcal{F}^{-1}\phi_k \mathcal{F} f(\cdot)| \Big| L_p(\mathbb{R}^n) \Big\|. \tag{1.2.10}$$

In what follows we will write $A^s_{p,q}(\mathbb{R}^n)$, where A stands either for B or F, if an assertion applies both to B- and F- spaces. A detailed study of these spaces including their history and properties can be found in [32], [33] and [35]. In particular they are independent (in the sense of equivalent quasi-norms) of the chosen resolution of unity as long as it fulfills (1.2.4) - (1.2.6). Therefore we will omit the subscript ϕ in Definition 1.2.1 in the sequel. We recall some special cases and basic properties, referring for further reading again to the above mentioned literature, in particular to [32, Section 2.3.8] and [35, Section 1.2].

(i) Let $\delta \in \mathbb{R}$. Then

$$I_\delta : \quad f \mapsto \mathcal{F}^{-1}((1+|\xi|)^{-\delta/2} \mathcal{F} f) \tag{1.2.11}$$

is a one-to-one map onto itself both in $S(\mathbb{R}^n)$ and in $S'(\mathbb{R}^n)$. Furthermore, I_δ is a lift for the spaces $A^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < p, q \leq \infty$ ($p < \infty$ for F-spaces), that is we have

$$I_\delta A^s_{p,q}(\mathbb{R}^n) = A^{s+\delta}_{p,q}(\mathbb{R}^n) \tag{1.2.12}$$

in the sense of equivalent quasi-norms.

(ii) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \tag{1.2.13}$$

is a well-known Littlewood-Paley theorem.

(iii) Applying I_δ with $\delta = s$ to (1.2.13) we obtain for $1 < p < \infty$, $s \in \mathbb{R}$ the Bessel-potential spaces

$$H^s_p(\mathbb{R}^n) = I_s L_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n). \tag{1.2.14}$$

(iv) If $s = k \in \mathbb{N}_0$ in (1.2.14) then

$$W^k_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n) \tag{1.2.15}$$

are the classical Sobolev spaces usually normed by

$$\|f\,|W^k_p(\mathbb{R}^n)\| = \Big(\sum_{|\gamma| \leq k} \|D^\gamma f|L_p(\mathbb{R}^n)\|^p \Big)^{1/p}. \tag{1.2.16}$$

(v) We denote the spaces

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n), \quad s \in \mathbb{R} \tag{1.2.17}$$

as Hölder-Zygmund spaces.

Remark 1.2.2. Note that the equalities in (ii) - (iv) has always to be understood in the sense of equivalent norms.

The above introduced inhomogeneous function spaces have some homogeneous counterparts. Let

$$\dot{S}(\mathbb{R}^n) = \{\phi \in S(\mathbb{R}^n) : \ D^\gamma(\mathcal{F}\phi)(0) = 0 \text{ for } \gamma \in \mathbb{N}_0^n\} \tag{1.2.18}$$

be the closed locally convex subspace of $S(\mathbb{R}^n)$ equipped with the same topology and $\dot{S}'(\mathbb{R}^n)$ its topological dual. Let again $\phi_0 \in S(\mathbb{R}^n)$ with

$$\phi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{ and } \quad \phi_0(x) = 0 \text{ if } |x| \ge 3/2 \tag{1.2.19}$$

and let

$$\phi^j(x) = \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n, \ j \in \mathbb{Z}. \tag{1.2.20}$$

Then

$$\sum_{j\in\mathbb{Z}} \phi^j(x) = 1 \quad \text{ for } \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.21}$$

Definition 1.2.3. Let $\phi = \{\phi^j\}_{j=-\infty}^{\infty}$ be the above homogeneous dyadic resolution of unity in $\mathbb{R}^n \setminus \{0\}$.

(i) For $0 < p, q \le \infty$, $s \in \mathbb{R}$ we define $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as the collection of all $f \in \dot{S}'(\mathbb{R}^n)$ such that

$$\|f|\dot{B}^s_{p,q}(\mathbb{R}^n)\|_\phi = \left(\sum_{j=-\infty}^{\infty} 2^{jsq}\|\mathcal{F}^{-1}\phi^j\mathcal{F}f|L_p(\mathbb{R}^n)\|^q\right)^{1/q} < \infty$$

with the usual modification if $q = \infty$.

(ii) For $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ the space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in \dot{S}'(\mathbb{R}^n)$ such that

$$\|f|\dot{F}^s_{p,q}(\mathbb{R}^n)\|_\phi = \| \left(\sum_{j=-\infty}^{\infty} 2^{jsq}|\mathcal{F}^{-1}\phi^j\mathcal{F}f(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\| < \infty$$

with the usual modification if $q = \infty$.

Remark 1.2.4. These spaces are again independent of ϕ in the sense of equivalent quasi-norms. They are based on the interpretation of $\dot{S}'(\mathbb{R}^n)$ as $S'(\mathbb{R}^n)/\mathcal{P}$, where $\mathcal{P}$ stands for the set of all polynomials in $\mathbb{R}^n$. For further details we refer to [39, pp. 17-20].

We define the vector-valued counterparts of the above function spaces.

Definition 1.2.5. Let $X(\mathbb{R}^n)$ be a (quasi-)normed function space. Then

$$X(\mathbb{R}^n)_n = \prod_{j=1}^{n} X(\mathbb{R}^n) \tag{1.2.22}$$

is the collection of all $f = (f_1, \ldots, f_n)$, $f_j \in X(\mathbb{R}^n)$ (quasi-)normed by

$$\|f|X(\mathbb{R}^n)_n\| = \sum_{j=1}^{n} \|f_j|X(\mathbb{R}^n)\|. \tag{1.2.23}$$

The corresponding definition for matrix-valued function spaces reads as follows.

Definition 1.2.6. Let $X(\mathbb{R}^n)$ be a (quasi-)normed function space. Then

$$X(\mathbb{R}^n)_{n\times n} = \prod_{i,j=1}^{n} X(\mathbb{R}^n) \tag{1.2.24}$$

is the collection of all $f = (f_1, \ldots, f_n)$, $f_j \in X(\mathbb{R}^n)_n$ (quasi-)normed by

$$\|f|X(\mathbb{R}^n)_{n\times n}\| = \sum_{i,j=1}^{n} \|f_{i,j}|X(\mathbb{R}^n)\|. \tag{1.2.25}$$

At the end of this section we introduce the solution spaces we are interested in. These are weighted Lebesgue spaces with respect to the Bochner integral.

Definition 1.2.7. Let X be a Banach space with $X \subset L_1^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < T < \infty$, $b \in \mathbb{R}$ and $1 \le v \le \infty$. Then $L_v((0,T),\, b,\, X)$ contains all $f : (0,T) \to X$ such that

$$\|f|L_v((0,T),\, b,\, X)\| = \left(\int_0^T t^{bv} \|f(\cdot,t)|X\|^v \mathrm{d}t\right)^{1/v} \tag{1.2.26}$$

is finite. If $v = \infty$ we replace (1.2.26) by

$$\|f|L_\infty((0,T),\, b,\, X)\| = \sup_{0<t<T} t^b \|f(\cdot,t)|X\|. \tag{1.2.27}$$

In the sequel we deal with Banach spaces $X(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n)$ and $X(\mathbb{R}^n)_n = A^s_{p,q}(\mathbb{R}^n)_n$, respectively, where A stands either for B or F, with $1 \le p, q \le \infty$ ($p < \infty$ for F-spaces).

Remark 1.2.8. After extending functions f belonging to $L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n))$ from $\mathbb{R}^n \times (0,T)$ to $\mathbb{R}^{n+1}$ by zero it holds $L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n)) \subset S'(\mathbb{R}^{n+1})$ if $b < 1 - 1/v$ or $b = 0$ and $v = \infty$, cf. [38, pp. 115, 116].

1.3 Decomposition methods

One of our main tools will be the characterization of the spaces $A^s_{p,q}(\mathbb{R}^n)$ by means of wavelets of Daubechies type and molecules. We recall the necessary definitions and assertions as far as we need them for our considerations. Standard references with respect to wavelets are e.g. [8], [25], [43]. For molecules we refer e.g. to [31], [15, Section 12], [16, Section 5].

1.3.1 Wavelets

As usual, $C^u(\mathbb{R})$, $u \in \mathbb{N}$, denotes the space of all complex-valued u- times continuously differentiable functions with bounded derivatives in $\mathbb{R}$. Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \tag{1.3.1}$$

be real-valued compactly supported Daubechies wavelets with $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ and

$$\int_{\mathbb{R}} x^v \psi_M(x)\mathrm{d}x = 0 \quad \text{for all } v \in \{0, \dots, u-1\}. \tag{1.3.2}$$

We always assume that ψ_F and ψ_M have L_2-norm 1. Then

$$\{\psi_F(x-m),\, 2^{j/2}\,\psi_M(2^j x - m) : \quad j \in \mathbb{N}_0,\, m \in \mathbb{Z}\} \tag{1.3.3}$$

is an orthonormal basis in $L_2(\mathbb{R})$ for any $u \in \mathbb{N}$. We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual multiresolution procedure. Let either

$$G = (G_1, \dots, G_n) \in G^0 = \{F,\, M\}^n \tag{1.3.4}$$

which means that G_r is either F or M or let

$$G = (G_1, \dots, G_n) \in G^j = \{F,\, M\}^{n*}, \quad j \in \mathbb{N}. \tag{1.3.5}$$

Here $*$ indicates that at least one of the components of G must be an M. In the sequel, we denote such a set G^j with G^*. We put

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^n \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j,\ m \in \mathbb{Z}^n, \tag{1.3.6}$$

$x \in \mathbb{R}^n$, now with $j \in \mathbb{N}_0$. Then for any $u \in \mathbb{N}$

$$\Psi = \{\Psi^j_{G,m} :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\} \tag{1.3.7}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m} =: \sum_{j,G,m} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m}$$

with

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x)\Psi^j_{G,m}(x)\mathrm{d}x = 2^{jn/2} \left\langle f, \Psi^j_{G,m} \right\rangle$$

is the corresponding expansion, adapted to our needs, where $2^{-jn/2}\Psi^j_{G,m}$ are uniformly bounded functions with respect to j and m. For more detailed explanations cf. [36, Subsection 1.2.1].

Remark 1.3.1. In particular, $\Psi^j_{G,m}$ satisfies the moment conditions if $G \in G^*$ since

$$\begin{aligned} \int_{\mathbb{R}} x^\nu \psi_M(2^j x - m)\mathrm{d}x &= 2^{-j(\nu+1)} \int_{\mathbb{R}} (y+m)^\nu \psi_M(y)\mathrm{d}y \\ &= 2^{-j(\nu+1)} \sum_{k=0}^{\nu} c_{\nu,k} \int_{\mathbb{R}} y^{\nu-k} m^k \psi_M(y)\mathrm{d}y \\ &= 0, \quad \text{for all } \nu < u. \end{aligned} \tag{1.3.8}$$

Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be cubes in $\mathbb{R}^n$ with side length 2^{-j+1} parallel to the coordinate axis and $2^{-j}m$ as left lower corner. Let $\chi_{j,m}$ denote their characteristic functions.

Definition 1.3.2. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Let

$$\lambda := \{\lambda_m^{j,G} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}. \tag{1.3.9}$$

Then

$$b^s_{p,q}(\mathbb{R}^n) = \{\lambda : \ \|\lambda | b^s_{p,q}(\mathbb{R}^n)\| < \infty\} \tag{1.3.10}$$

with

$$\|\lambda | b^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{1.3.11}$$

and

$$f^s_{p,q}(\mathbb{R}^n) = \{\lambda : \ \|\lambda | f^s_{p,q}(\mathbb{R}^n)\| < \infty\} \tag{1.3.12}$$

with

$$\|\lambda | f^s_{p,q}(\mathbb{R}^n)\| = \left\| \left(\sum_{j,G,m} 2^{jsq} |\lambda_m^{j,G} \chi_{j,m}|^q \right)^{\frac{1}{q}} |L_p(\mathbb{R}^n) \right\| \tag{1.3.13}$$

(usual modifications if $\max(p,q) = \infty$).

The following decomposition theorem may be found in [36, Theorem 1.20, pp.15-17]. Let σ_p and $\sigma_{p,q}$ be as defined in (1.1.1).

Theorem 1.3.3. **(i)** *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* $\{\Psi^j_{G,m}\}$ *be the wavelet system in* (1.3.7) *based on* (1.3.1) *and* (1.3.2) *with*

$$u > \max(s, \sigma_p - s). \tag{1.3.14}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in b^s_{p,q}(\mathbb{R}^n), \tag{1.3.15}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. *The representation is unique,*

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \left\langle f, \Psi^j_{G,m} \right\rangle \tag{1.3.16}$$

and

$$I: \quad f \mapsto \{\lambda^{j,G}_m(f)\} \tag{1.3.17}$$

is an isomorphic map from $B^s_{p,q}(\mathbb{R}^n)$ *onto* $b^s_{p,q}(\mathbb{R}^n)$. *In particular, it holds* $\|f\,|B^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda\,|b^s_{p,q}(\mathbb{R}^n)\|$.

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_{p,q} - s). \tag{1.3.18}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in f^s_{p,q}(\mathbb{R}^n), \tag{1.3.19}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. *The representation* (1.3.19) *is unique with* (1.3.16). *Furthermore* I *in* (1.3.17) *is an isomorphic map from* $F^s_{p,q}(\mathbb{R}^n)$ *onto* $f^s_{p,q}(\mathbb{R}^n)$ *and* $\|f\,|F^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda\,|f^s_{p,q}(\mathbb{R}^n)\|$.

Remark 1.3.4. For a detailed discussion how to understand $\left\langle f, \Psi^j_{G,m} \right\rangle$ as a dual pairing of $f \in A^s_{p,q}(\mathbb{R}^n)$ and $\Psi^j_{G,m} \in C^u(\mathbb{R}^n)$ we refer to [35, Section 3.1.3] and for further reading to [32, Section 2.11.1 - 2.11.3].

1.3.2 Molecules

Next we recall the molecular counterpart of Theorem 1.3.3. Therefore, we have to introduce corresponding sequence spaces which differ from those for wavelets only by the lack of the finite sum over G.

Definition 1.3.5. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Let

$$\mu := \{\mu_m^j \in \mathbb{C} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\}. \tag{1.3.20}$$

Then

$$\bar{b}^s_{p,q}(\mathbb{R}^n) = \{\mu : \ \|\mu \,|\, \bar{b}^s_{p,q}(\mathbb{R}^n)\| < \infty\} \tag{1.3.21}$$

with

$$\|\mu \,|\, \bar{b}^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} |\mu_m^j|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \tag{1.3.22}$$

and

$$\bar{f}^s_{p,q}(\mathbb{R}^n) = \{\mu : \ \|\mu|\bar{f}^s_{p,q}(\mathbb{R}^n)\| < \infty\} \tag{1.3.23}$$

with

$$\|\mu \,|\, \bar{f}^s_{p,q}(\mathbb{R}^n)\| = \left\| \left(\sum_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} 2^{jsq} |\mu_m^j \chi_{j,m}(\cdot)|^q \right)^{\frac{1}{q}} |L_p(\mathbb{R}^n) \right\| \tag{1.3.24}$$

(usual modifications if $\max(p, q) = \infty$).

Definition 1.3.6. Let $K \in \mathbb{N}_0$, $N \in \mathbb{N}_0$, and $L > N + n - 1$. Then L_∞-functions $b_{j,m} : \mathbb{R}^n \mapsto \mathbb{C}$ are called (K, N, L)- molecules, related to $Q_{j,m}$, if

$$|D^\gamma b_{j,m}(x)| \leq 2^{j|\gamma|}(1 + 2^j|x - 2^{-j}m|)^{-L}, \quad |\gamma| \leq K, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.3.25}$$

and

$$\int_{\mathbb{R}^n} x^\beta b_{j,m}(x) \, \mathrm{d}x = 0, \quad |\beta| < N, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{1.3.26}$$

Remark 1.3.7. (i) If $N = 0$ then (1.3.26) is empty (no condition).

(ii) If $K = 0$ then $b_{j,m}$ need not to be continuous. For $K \in \mathbb{N}$ we assume that the classical derivatives exist up to order K inclusively and are continuous.

(iii) The condition $L > N + n - 1$ ensures the existence of (1.3.26) as L_1 - integral as a substitute for the lack of a compact support.

The following analogue of Proposition 1.3.3 for molecules can be found in [31]. Similar to our notation for the $A^s_{p,q}(\mathbb{R}^n)$ - spaces we write $\bar{a}^s_{p,q}(\mathbb{R}^n)$ and $a^s_{p,q}(\mathbb{R}^n)$ with $a \in \{b, f\}$.

Theorem 1.3.8. *Let $0 < p, q \leq \infty$ ($p < \infty$ for the F-spaces), $s \geq 0$. Let $K \in \mathbb{N}_0$, $N \in \mathbb{N}_0$ with*

$$K > s, \quad N > \begin{cases} \sigma_p - s, & B\text{-spaces}, \\ \sigma_{p,q} - s, & F\text{-spaces} \end{cases} \tag{1.3.27}$$

and

$$L > N + n - 1, \quad L > n + \begin{cases} \sigma_p, & B\text{-spaces}, \\ \sigma_{p,q}, & F\text{-spaces}. \end{cases} \tag{1.3.28}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in A^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \mu^j_m b_{j,m}, \quad \mu \in \bar{a}^s_{p,q}(\mathbb{R}^n), \tag{1.3.29}$$

unconditional convergence being in $S'(\mathbb{R}^n)$, where $b_{j,m}$ are (N, K, L)-molecules. Furthermore

$$\|f | A^s_{p,q}(\mathbb{R}^n)\| \sim \inf \|\mu \mid \bar{a}^s_{p,q}(\mathbb{R}^n)\| \tag{1.3.30}$$

where the infimum is taken over all admissible representations.

1.4 Embeddings and multiplication properties

In connection with nonlinear generalized heat equations and generalized Navier-Stokes equations one has to deal with mapping properties of type $u \mapsto u^2$ in the spaces $A^s_{p,q}(\mathbb{R}^n)$. More precisely we ask for conditions on the spaces $A^s_{p,q}(\mathbb{R}^n)$ such that the product $f_1 f_2$ of two elements $f_1,\ f_2 \in A^s_{p,q}(\mathbb{R}^n)$ belongs at least to a space $A^{\tilde{s}}_{\tilde{p},\tilde{q}}(\mathbb{R}^n)$. We recall first under which conditions $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^{\text{loc}}_1(\mathbb{R}^n)$ and $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$, respectively, holds where $\hookrightarrow$ stands for the continuous embedding, cf. [30, Theorem 3.3.2 and Corollary 3.3.1].

Theorem 1.4.1. **(i)** *The following two assertions are equivalent:*

(a) $F^s_{p,q}(\mathbb{R}^n) \subset L^{loc}_1(\mathbb{R}^n)$,

(b) *either* $0 < p < 1, \quad s \geq \sigma_p, \quad 0 < q \leq \infty,$
or $1 \leq p < \infty, \quad s > 0, \quad 0 < q \leq \infty,$
or $1 \leq p < \infty, \quad s = 0, \quad 0 < q \leq 2.$

(ii) *The following two assertions are equivalent:*

(a) $B^s_{p,q}(\mathbb{R}^n) \subset L_1^{loc}(\mathbb{R}^n)$,

(b) *either* $0 < p \le \infty, \quad s > \sigma_p, \quad 0 < q \le \infty,$
or $0 < p \le 1, \quad s = n\left(\frac{1}{p} - 1\right), \quad 0 < q \le 1,$
or $1 < p \le \infty, \quad s = 0, \quad 0 < q \le \min(p, 2)$.

Corollary 1.4.2. *Let $p < \infty$. Then the following two assertions are equivalent*

(a) $A^s_{p,q}(\mathbb{R}^n) \subset L_1^{loc}(\mathbb{R}^n)$,

(b) $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_{\overline{p}}(\mathbb{R}^n), \ \overline{p} = \max(1, p)$.

The next two subsections summarize the corresponding multiplication and embedding results. First we focus on the case when $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra, i.e. $f_1 f_2 \in A^s_{p,q}(\mathbb{R}^n)$ if $f_1 \in A^s_{p,q}(\mathbb{R}^n)$ and $f_2 \in A^s_{p,q}(\mathbb{R}^n)$. Provided that we do not have a multiplication algebra our proofs rely on Hölder inequalities.

1.4.1 Multiplication algebras

Recall that $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n)$ is called a multiplication algebra if $f_1 f_2 \in A^s_{p,q}(\mathbb{R}^n)$ for any $f_1, \, f_2 \in A^s_{p,q}(\mathbb{R}^n)$ and if there is a constant $c > 0$ such that

$$\|f_1 f_2 | A^s_{p,q}(\mathbb{R}^n)\| \le c \|f_1 | A^s_{p,q}(\mathbb{R}^n)\| \|f_2 | A^s_{p,q}(\mathbb{R}^n)\| \tag{1.4.1}$$

for all $f_1, \, f_2 \in A^s_{p,q}(\mathbb{R}^n)$. The following theorem may be found in [29, Theorem 6.4.1].

Theorem 1.4.3. *Let s>0.*

(i) *The following statements are equivalent:*

(a) $F^s_{p,q}(\mathbb{R}^n)$ *is a multiplication algebra,*

(b) $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$,

(c) *either $s > n/p$ or $s = n/p$ and $0 < p \le 1$.*

(ii) *The following statements are equivalent:*

(a) $B^s_{p,q}(\mathbb{R}^n)$ *is a multiplication algebra,*

(b) $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$,

(c) *either $s > n/p$ or $s = n/p$ and $0 < q \le 1$.*

1.4.2 Hölder inequalities and useful embeddings

The next assertions based on [30, Theorems 4.2.1 and 4.4.1] are already adapted to our needs.

Theorem 1.4.4. *Let $s > 0$, $0 < p_1, p < \infty$ and $0 < q \leq \infty$. Let*

$$\frac{1}{r} = \frac{1}{p_1} - \frac{s}{n} > 0 \quad \textit{and} \quad 0 < \frac{2}{r} = \frac{1}{p} - \frac{s}{n} < 1.$$

(i) *Then holds*

$$B^s_{p_1,q}(\mathbb{R}^n) \cdot B^s_{p_1,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n)$$

if, and only if,

$$0 < q \leq r.$$

(ii) *Then holds*

$$F^s_{p_1,q}(\mathbb{R}^n) \cdot F^s_{p_1,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n).$$

Theorem 1.4.5. *Let $1 \leq p_1, p < \infty$, $1 \leq q \leq \infty$ such that*

$$\frac{1}{p_1} < \frac{1}{p} < \frac{2}{p_1} \tag{1.4.2}$$

$$2s > \max\left(0, n\left(\frac{2}{p_1} - 1\right)\right) \tag{1.4.3}$$

$$s > n\left(\frac{2}{p_1} - \frac{1}{p}\right). \tag{1.4.4}$$

Then

$$A^s_{p_1,q}(\mathbb{R}^n) \cdot A^s_{p_1,q}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(\mathbb{R}^n). \tag{1.4.5}$$

Starting with elements in $A^s_{p,q}(\mathbb{R}^n)$ the above Hölder inequalities give estimates in some spaces $A^{\tilde{s}}_{\tilde{p},\tilde{q}}(\mathbb{R}^n)$. To return to our initial space $A^s_{p,q}(\mathbb{R}^n)$, we need the following embedding, cf. [30, Theorem 3.2.1], already adapted to our needs, too.

Theorem 1.4.6. *Let $0 < p < p_1 \leq \infty$ ($p_1 < \infty$ for F-spaces), s, $s_1 \in \mathbb{R}$ with $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$ and $0 < u, q \leq \infty$.*

(i) *Then*

$$B^s_{p,u}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,q}(\mathbb{R}^n)$$

if, and only if,

$$0 < u \leq q \leq \infty.$$

(ii) *For F-spaces*

$$F^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^n)$$

holds.

2 Generalized heat equations

In this chapter we prepare the construction of solutions of the generalized nonlinear heat equation, the generalized Navier-Stokes equations and its reformulated version. As indicated in the introduction they will be represented by means of the convolution with the kernel G^α, $\alpha \in \mathbb{N}$ given by

$$G^\alpha(x,t) = \begin{cases} \frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}} \right)^\vee (x), & t > 0,\, x \in \mathbb{R}^n, \\ 0, & t \leq 0,\ x \in \mathbb{R}^n. \end{cases}$$

For this purpose we recall the basic facts concerning tensor products and convolutions in $D'(\mathbb{R}^n)$. Then we show that G^α is a fundamental solution of the generalized heat equation, i.e. it holds

$$(\partial_t + (-\Delta_x)^\alpha) G^\alpha = \delta$$

where δ denotes the δ- distribution and deal with the Cauchy problem of the inhomogeneous generalized heat equation in the distributional sense. Finally, we provide Fourier analytical methods which we need later on in Chapter 5 when it comes to generalized Navier-Stokes equations and its reformulation. In particular we introduce the so-called Leray projector defined by means of scalar Riesz transforms and recall its basic mapping properties in spaces of Besov and Triebel-Lizorkin type.

2.1 Tensor product and convolution

First we recall the definition of the tensor product of two distributions $f \in D'(\mathbb{R}^n)$ and $g \in D'(\mathbb{R}^m)$ and some of its basic properties. We follow the representation in [34] and [42].

Theorem 2.1.1. *Let $f \in D'(\mathbb{R}^n)$ and $g \in D'(\mathbb{R}^m)$. Then there exists exactly one distribution $h \in D'(\mathbb{R}^{n+m})$ such that for all functions $\phi(x) \in D(\mathbb{R}^n)$ and $\psi(y) \in D(\mathbb{R}^m)$ holds*

$$h(\phi(x)\psi(y)) = f(\phi(x))\, g(\psi(y)). \tag{2.1.1}$$

For $\rho(x,y) \in D(\mathbb{R}^{n+m})$ we have

$$h(\rho(x,y)) = f_x(g_y(\rho(x,y))) = g_y(f_x(\rho(x,y))). \tag{2.1.2}$$

Definition 2.1.2. Let $f \in D'(\mathbb{R}^n)$ and $g \in D'(\mathbb{R}^m)$. The distribution $h \in D'(\mathbb{R}^{n+m})$, given by (2.1.1) and shown to be unique by the above theorem, is called the tensor product of f and g, being denoted by $h = f \otimes g$.

The theorem summarizes some properties of the tensor product.

Theorem 2.1.3. *Let $f \in D'(\mathbb{R}^n)$, $g \in D'(\mathbb{R}^m)$, and let $h \in D'(\mathbb{R}^l)$. Then*

$$f_x \otimes g_y = g_y \otimes f_x \quad \textit{(commutativity)}$$

and

$$(f_x \otimes g_y) \otimes h_z = f_x \otimes (g_y \otimes h_z) \quad \textit{(associativity).}$$

Further,

$$D^\gamma (f_x \otimes g_y) = (D^\gamma f_x) \otimes g_y$$

and

$$\operatorname{supp}(f \otimes g) = \{(x, y) : \ (x, y) \in \mathbb{R}^{n+m},\ x \in \operatorname{supp} f,\ y \in \operatorname{supp} g\}.$$

The tensor product is continuous that means if the distributions f_k, $k \in \mathbb{N}$, belong to $D'(\mathbb{R}^n)$ and if $f_k(\phi) \overset{k\to\infty}{\longrightarrow} f(\phi)$ for every function $\phi \in D(\mathbb{R}^n)$ then

$$(f_k \otimes g)(\rho) \overset{k\to\infty}{\longrightarrow} (f \otimes g)(\rho)$$

for every function $\rho \in D(\mathbb{R}^{n+m})$.

The proofs of the above theorem as well as the definition may be found in [34, Sections 3.1.2, 3.1.3].
For our later considerations we also need the convolution of distributions in $D'(\mathbb{R}^n)$. Here we follow [42, Sections 4.1, 4.2]. Let f and g be locally integrable functions in $\mathbb{R}^n$. If the integral $\int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}y$ exists almost everywhere in $\mathbb{R}^n$ and defines a locally integrable function then it is called the convolution of the functions f and g. It is denoted by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}y = \int_{\mathbb{R}^n} g(y)f(x-y)\mathrm{d}y = (g * f)(x). \tag{2.1.3}$$

The convolution $f * g$ is a regular distribution and can be represented by

$$(f * g)(\phi) = \int_{\mathbb{R}^{2n}} f(x)g(y)\phi(x+y)\mathrm{d}x\,\mathrm{d}y, \quad \forall \phi \in D(\mathbb{R}^n). \tag{2.1.4}$$

Assume now that the sequence $\{\eta_k\}_k \subset D(\mathbb{R}^n)$ has the following properties:

(i) for any compact subset $K \subset \mathbb{R}^n$ there exists a number $N \in \mathbb{N}$ such that $\eta_k(x) = 1$ for all $x \in K$ and $k \geq N$,

(ii) for any multi-index $\gamma \in \mathbb{N}_0^n$ there exists a constant c_γ such that $|D^\gamma \eta_k(x)| < c_\gamma$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

Such a sequence can be easily constructed by taking a test function $\eta \in D(\mathbb{R}^n)$ with $\eta(x) = 1$ if $|x| < 1$ and defining $\eta_k(x) := \eta(x/k)$. Then the convolution of two distributions in $D'(\mathbb{R}^n)$ is defined as follows

Definition 2.1.4. Let f, $g \in D'(\mathbb{R}^n)$ and $\{\eta_k\}_k \subset D(\mathbb{R}^{2n})$ be the above defined sequence. Then the convolution of f and g is defined by

$$(f * g)(\phi) := \lim_{k\to\infty} (f \otimes g)(\eta_k(x, y)\phi(x + y)) \tag{2.1.5}$$

if the limit exists and is independent of the choice of $\{\eta_k\}_k$.

Remark 2.1.5. Then $f * g$ belongs to $D'(\mathbb{R}^n)$. Further we have that if f, $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and if $\int_{\mathbb{R}^n} f(y)g(x - y)\mathrm{d}y \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ then there exists $(f * g) \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and coincides with (2.1.3) in the sense of (2.1.4).

Example 2.1.6. Let f, $g \in D'(\mathbb{R}^n)$ and $\phi \in D(\mathbb{R}^n)$ such that the set

$$\{(x, y) : \ (x, y) \in \mathbb{R}^{2n},\ x \in \operatorname{supp} f,\ y \in \operatorname{supp} g,\ x + y \in \operatorname{supp} \phi\} \tag{2.1.6}$$

is bounded for all $\phi \in D(\mathbb{R}^n)$. Then $f * g$ exists. This holds in particular if one of the distributions has compact support. Thus the convolution of any $f \in D'(\mathbb{R}^n)$ with the δ-distribution δ exists and it holds

$$f * \delta = \delta * f = f.$$

Example 2.1.7. (Young's inequality) Let $f \in L_p(\mathbb{R}^n)$, $g \in L_q(\mathbb{R}^n)$ and p, q, $r \geq 1$. Then $f * g \in L_r(\mathbb{R}^n)$ and

$$\|f * g|L_r(\mathbb{R}^n)\| \leq \|f|L_p(\mathbb{R}^n)\|\, \|g|L_q(\mathbb{R}^n)\| \qquad \text{if } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

We summarize some properties according to [42, Sections 4.2.1, 4.2.5, 4.2.7] where one can also find the proofs.

Theorem 2.1.8. *Let f, $g \in D'(\mathbb{R}^n)$ be such that $f * g$ exists. Then it holds*

(i) $f * g = g * f$,

(ii) $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$.

(iii) *Let $\gamma \in \mathbb{N}_0^n$ be a multi-index. Then there exist $(D^\gamma f) * g$ and $f * (D^\gamma g)$ and it holds*

$$D^\gamma(f * g) = (D^\gamma f) * g = f * (D^\gamma g).$$

2.2 Fundamental solution

In this section we specify a fundamental solution of the generalized heat equation and show that it is a regular distribution. Recall that $G^\alpha \in D'(\mathbb{R}^{n+1})$, $\alpha \in \mathbb{N}$ is called a fundamental solution of the generalized heat equation if it satisfies

$$(\partial_t + (-\Delta_x)^\alpha)G^\alpha = \delta \tag{2.2.1}$$

where $\delta \in D'(\mathbb{R}^{n+1})$ is the δ- distribution.

Proposition 2.2.1. *Let $\alpha \in \mathbb{N}$. Then the regular distribution*

$$G^\alpha(x,t) = \begin{cases} \frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee (x) & t > 0,\ x \in \mathbb{R}^n, \\ 0 & t \leq 0,\ x \in \mathbb{R}^n \end{cases} \tag{2.2.2}$$

where the inverse Fourier transform is taken with respect to ξ, is a fundamental solution of the generalized heat equation in $D'(\mathbb{R}^{n+1})$.

Proof. Step 1. To prove that $G^\alpha(x,t)$ is a regular distribution we show that it is integrable on the set $M = \{(x,t) : x \in \mathbb{R}^n,\ 0 < t < t_0\}$ and obtain

$$\begin{aligned} \int_M |G^\alpha(x,t)| \mathrm{d}x\,\mathrm{d}t &\leq \frac{1}{(2\pi)^{n/2}} \int_0^{t_0} \left| \int_{\mathbb{R}^n} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee (x)\,\mathrm{d}x \right| \mathrm{d}t \\ &= \frac{1}{(2\pi)^{n/2}} \int_0^{t_0} t^{-\frac{n}{2\alpha}} \left| \int_{\mathbb{R}^n} \left(e^{-|\eta|^{2\alpha}}\right)^\vee \left(\frac{x}{t^{1/2\alpha}}\right) \mathrm{d}x \right| \mathrm{d}t \\ &= \frac{1}{(2\pi)^{n/2}} \int_0^{t_0} \left| \int_{\mathbb{R}^n} \left(e^{-|\eta|^{2\alpha}}\right)^\vee (y)\,\mathrm{d}y \right| \mathrm{d}t \\ &= \frac{t_0}{(2\pi)^{n/2}} \left| \int_{\mathbb{R}^n} \left(e^{-|\eta|^{2\alpha}}\right)^\vee (y) e^{-i0\cdot y}\,\mathrm{d}y \right| = t_0. \end{aligned}$$

Step 2. We denote temporarily the Fourier transform of $\phi(x,t)$ with respect to $x \in \mathbb{R}^n$ by

$$(\mathcal{F}_x\phi)(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi}\phi(x,t)\,\mathrm{d}x, \quad \phi \in S(\mathbb{R}^{n+1}).$$

According to [34, Theorem 3.2.4/1], $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ provide bijective mappings on both $S(\mathbb{R}^{n+1})$ and $S'(\mathbb{R}^{n+1})$. In the sequel we write $G_t^\alpha(x)$ when we fix $t \in (0,\infty)$. We have to verify that $G^\alpha \in S'(\mathbb{R}^{n+1})$. With

$$\|\phi\|_{k,l} = \sup_{(x,t)\in\mathbb{R}^{n+1}} (1+|x|+|t|)^k \sum_{|\gamma|\leq l} |D^\gamma\phi(x,t)|, \quad k,l \in \mathbb{N}_0 \tag{2.2.3}$$

it holds that

$$\begin{aligned}
|G^\alpha(\phi)| &\leq \int_{\mathbb{R}^n}\int_0^\infty \left|\left(e^{-t|\xi|^{2\alpha}}\right)^\vee(x)\phi(x,t)\right| \mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n}\int_0^\infty \left|\left(e^{-|\eta|^{2\alpha}}\right)^\vee(y)\phi(t^{1/2\alpha}y,t)\right| \mathrm{d}t\,\mathrm{d}y \\
&\leq \|\phi\|_{k,0}\int_{\mathbb{R}^n}\int_0^\infty \frac{1}{1+|y|^m}\,\frac{1}{1+t^{k/2\alpha}|y|^k+t^k}\mathrm{d}t\,\mathrm{d}y \\
&\leq \|\phi\|_{k,0}\int_{\mathbb{R}^n}\int_0^\infty \frac{1}{1+|y|^m}\,\frac{1}{1+t^k}\mathrm{d}t\,\mathrm{d}y, \qquad k,m\in\mathbb{N}_0 \\
&\leq c_{m,k}\|\phi\|_{k,0}
\end{aligned}$$

if $k \geq 2$ and $m \geq n+1$. Hence G^α is a regular distribution belonging to $S'(\mathbb{R}^{n+1})$ and we can write for $\phi \in S(\mathbb{R}^{n+1})$

$$G^\alpha(\mathcal{F}_x\phi) = \int_0^\infty G_t^\alpha(\mathcal{F}_x\phi)(t)\mathrm{d}t = \int_0^\infty (\mathcal{F}_xG_t^\alpha)(\phi(x,t))\mathrm{d}t. \tag{2.2.4}$$

From the explicit form of G^α it follows that $G_t^\alpha \in S(\mathbb{R}^n)$. Thus

$$\begin{aligned}
(\mathcal{F}_xG_t^\alpha)(\xi,t) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\xi}\left(\frac{1}{t^{n/2\alpha}}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i\frac{x}{t^{1/2\alpha}}\eta}e^{-|\eta|^{2\alpha}}\mathrm{d}\eta\right)\mathrm{d}x \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-iyt^{1/2\alpha}\xi}\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{iy\eta}e^{-|\eta|^{2\alpha}}\mathrm{d}\eta\right)\mathrm{d}y \\
&= \frac{1}{(2\pi)^{n/2}}e^{-t|\xi|^{2\alpha}}.
\end{aligned}$$

Inserting this into (2.2.4) we get

$$G^\alpha(\mathcal{F}_x\phi) = \frac{1}{(2\pi)^{n/2}}\int_0^\infty\int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}}\phi(\xi,t)\mathrm{d}\xi\,\mathrm{d}t. \tag{2.2.5}$$

Hence,

$$\begin{aligned}
&(\partial_tG^\alpha + (-\Delta_x)^\alpha G^\alpha)(\mathcal{F}_x\phi) = G^\alpha(\mathcal{F}_x(-\partial_t\phi + |x|^{2\alpha}\phi)) \\
&= \frac{1}{(2\pi)^{n/2}}\int_0^\infty\int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}}(-\partial_t\phi(\xi,t) + |\xi|^{2\alpha}\phi(\xi,t))\mathrm{d}\xi\,\mathrm{d}t \\
&= -\frac{1}{(2\pi)^{n/2}}\int_0^\infty\int_{\mathbb{R}^n} \partial_t(e^{-t|\xi|^{2\alpha}}\phi(\xi,t))\mathrm{d}\xi\,\mathrm{d}t \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\phi(\xi,0)\mathrm{d}\xi = (\mathcal{F}_x\phi)(0,0) = \delta(\mathcal{F}_x\phi).
\end{aligned}$$

Here we have used Fubinis theorem and (2.2.5). Since $\mathcal{F}_x$ is a bijective mapping on $S(\mathbb{R}^{n+1})$ this leads to

$$(\partial_t + (-\Delta_x)^\alpha)G^\alpha = \delta.$$

□

2.3 Cauchy's problem

We shall now use the results of the last two sections to formulate and treat Cauchy's problem for an inhomogeneous generalized heat equation. We do not specify the inhomogeneity yet since its concrete form is not necessary for the rather general results presented in this part. The ideas follow those of [34, Section 3.3.4]. Let $\alpha \in \mathbb{N}$. The classical Cauchy problem can be described as follows. Let $f(x,t)$ with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ be a function which is continuous on the half-space $\overline{\mathbb{R}^{n+1}_+}$ and $u_0(x)$ a function which is continuous on $\mathbb{R}^n$. We look for a function $u(x,t)$, 2α-times continuously differentiable on the domain $\mathbb{R}^{n+1}_+$ and continuous on the half-space $\overline{\mathbb{R}^{n+1}_+}$, that satisfies

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) &= f(x,t), \qquad x \in \mathbb{R}^n,\ t > 0, && (2.3.1)\\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}^n. && (2.3.2) \end{aligned}$$

A more general formulation of the Cauchy problem in $D'(\mathbb{R}^{n+1}_+)$ is given as follows. Let $U_0 \in D'(\mathbb{R}^n)$ and δ be the δ- distribution in $\mathbb{R}$. Further, let $F \in D'(\mathbb{R}^{n+1})$ with $\operatorname{supp} F \subset \overline{\mathbb{R}^{n+1}_+}$. Then $U \in D'(\mathbb{R}^{n+1})$ with $\operatorname{supp} U \subset \overline{\mathbb{R}^{n+1}_+}$ and

$$\partial_t U + (-\Delta_x)^\alpha U = F + U_{0,x} \otimes \delta_t \tag{2.3.3}$$

is called a solution of the Cauchy problem in $\mathbb{R}^{n+1}$ with the initial data U_0. We show first that (2.3.3) is indeed a generalization of (2.3.1), (2.3.2).

Proposition 2.3.1. *Let $\alpha \in \mathbb{N}$. Let U_0, U, and F in* (2.3.3) *be regular distributions. Furthermore we assume that $U_0 = u_0(x)$ is continuous on $\mathbb{R}^n$, $F = f(x,t)$ is continuous on the half space $\overline{\mathbb{R}^{n+1}_+}$, and $U = u(x,t)$ is 2α-times continuously differentiable on the domain $\mathbb{R}^{n+1}_+$ and continuous on the half space $\overline{\mathbb{R}^{n+1}_+}$. Then the solution $u(x,t)$ of* (2.3.3) *is also a solution of* (2.3.1), (2.3.2)*. Conversely, every solution of the classical Cauchy problem* (2.3.1), (2.3.2) *is also a solution of* (2.3.3) *if the functions $f(x,t)$ and $u(x,t)$ are extended by zero into the domain $\{(x,t) : (x,t) \in \mathbb{R}^{n+1},\ t < 0\}$.*

Proof. Step 1. Let $\phi(x,t) \in D(\mathbb{R}^{n+1})$. If U is a solution of (2.3.3) then under the above assumptions

$$\begin{aligned} &(\partial_t U + (-\Delta_x)^\alpha U)(\phi) = F(\phi) + (U_{0,x} \otimes \delta_t)(\phi) \\ &= \int_0^\infty \int_{\mathbb{R}^n} f(x,t)\phi(x,t)\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} u_0(x)\phi(x,0)\mathrm{d}x. \end{aligned} \tag{2.3.4}$$

On the other hand we obtain by means of integration by parts

$$
\begin{aligned}
&(\partial_t U + (-\Delta_x)^\alpha U)(\phi) = U(-\partial_t\phi + (-\Delta_x)^\alpha \phi) \\
&= -\int_{\mathbb{R}^n}\int_0^\infty u(x,t)\partial_t\phi(x,t)\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}^n}\int_0^\infty u(x,t)(-\Delta_x)^\alpha\phi(x,t)\mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} u(x,0)\phi(x,0)\mathrm{d}x + \int_{\mathbb{R}^n}\int_0^\infty \big(\partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t)\big)\phi(x,t)\mathrm{d}t\,\mathrm{d}x
\end{aligned}
\tag{2.3.5}
$$

Comparing (2.3.4) and (2.3.5) it follows for $\phi \in D(\mathbb{R}^{n+1})$ with $\operatorname{supp}\phi \subset \mathbb{R}^{n+1}_+$ by the fundamental lemma of calculus of variations cf. e.g. [34, Lemma 1.4.2] that

$$
f(x,t) = \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t), \qquad x \in \mathbb{R}^n,\ t > 0.
$$

As for the initial data we choose $\eta \in D(\mathbb{R})$ with $\eta(t) = 1$ if $|t| \leq 1$ and $\psi \in D(\mathbb{R}^n)$. Then we set $\phi(x,t) = \eta(t)\psi(x)$. In combination with (2.3.4) and (2.3.5) this choice of ϕ yields

$$
\int_{\mathbb{R}^n} (u(x,0) - u_0(x))\psi(x) = 0
$$

such that we can conclude again by [34, Lemma 1.4.2] that $u(x,0) = u_0(x)$.
Step 2. Conversely, assume that $u(x,t)$ is a solution of (2.3.1), (2.3.2). Then by partial integration we obtain (2.3.5). Comparison with (2.3.4) shows that $U = u(x,t)$ for $t \geq 0$ and $U = 0$ for $t < 0$ is a solution in the distributional sense, i.e. of (2.3.3), with $F = f(x,t)$ for $t \geq 0$ and $F = 0$ for $t < 0$, and that $U_0 = u_0(x)$. □

We recall [34, Theorem 3.2.4/2] already adapted to our needs which gives us a solution of an inhomogeneous generalized heat equation in terms of the fundamental solution.

Proposition 2.3.2. *Let G^α be the fundamental solution obtained in Proposition 2.2.1 and $F \in D'(\mathbb{R}^{n+1})$ such that the convolution of F and G^α in the sense of Definition 2.1.4 exists. Then it holds*

$$
(\partial_t + (-\Delta_x)^\alpha)(G^\alpha * F) = F.
$$

Definition 2.3.3. Let $\omega \in S'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}$. Then we define

$$
W_t^\alpha\omega(x) := (G_t^\alpha * \omega)(x), \quad t > 0,\, x \in \mathbb{R}^n \tag{2.3.6}
$$

where $*$ denotes the convolution in $S'(\mathbb{R}^n)$.

Note that since $G_t^\alpha \in S(\mathbb{R}^n)$ the convolution exists for all $\omega \in S'(\mathbb{R}^n)$. Furthermore $W_t^\alpha\omega \in C^\infty(\mathbb{R}^n)$.

Proposition 2.3.4. *Let $\omega \in S'(\mathbb{R}^n) \cap L_1^{loc}(\mathbb{R}^n)$. Let $F \in L_1^{loc}(\mathbb{R}^{n+1}) \subset D'(\mathbb{R}^{n+1})$ with*

$$F(x,t) = \begin{cases} f(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ 0, & x \in \mathbb{R}^n,\ t \leq 0 \end{cases}$$

and suppose that its convolution with G^α in $D'(\mathbb{R}^{n+1})$ in the sense of Definition 2.1.4 exists. Further assume that

$$H(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} G^\alpha(t-\tau, x-y) F(y,\tau)\, dy\, d\tau = \begin{cases} h(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ 0, & x \in \mathbb{R}^n,\ t \leq 0 \end{cases}$$

with

$$h(x,t) = \int_0^t \int_{\mathbb{R}^n} G^\alpha(t-\tau, x-y) f(y,\tau)\, dy\, d\tau = \int_0^t W^\alpha_{t-\tau} f(x,\tau) d\tau \tag{2.3.7}$$

belongs to $L_1^{loc}(\mathbb{R}^{n+1})$. Then it holds (after extending $W_t^\alpha \omega$ to $\mathbb{R}^{n+1}$ by zero)

$$(\partial_t + (-\Delta_x)^\alpha)(W_t^\alpha \omega + H) = F \quad \text{in } D'(\mathbb{R}^{n+1}) \tag{2.3.8}$$

and

$$(\partial_t + (-\Delta_x)^\alpha)(W_t^\alpha \omega + h) = f \quad \text{in } D'(\mathbb{R}^{n+1}_+), \tag{2.3.9}$$

respectively.

Proof. Step 1. We verify that for any $\omega \in S'(\mathbb{R}^n) \cap L_1^{\mathrm{loc}}(\mathbb{R}^n)$

$$\big(\partial_t + (-\Delta_x)^\alpha\big) W_t^\alpha \omega(x) = 0, \quad x \in \mathbb{R}^n,\, t > 0 \tag{2.3.10}$$

in the classical sense and thus in $D'(\mathbb{R}^{n+1}_+)$. Since $G_t^\alpha \in S(\mathbb{R}^n)$ we have

$$\mathcal{F}_x\big(\partial_t + (-\Delta_x)^\alpha\big)\mathcal{F}_x^{-1}\big(e^{-t|\xi|^{2\alpha}}\big) = \big(\partial_t + |\xi|^{2\alpha}\big)e^{-t|\xi|^{2\alpha}} = 0,$$

that is

$$(-\Delta_x)^\alpha G^\alpha(x,t) = -\partial_t G^\alpha(x,t), \quad t > 0,\, x \in \mathbb{R}^n. \tag{2.3.11}$$

Hence, G^α is a classical solution in $\mathbb{R}^{n+1}_+$ and thus in $D'(\mathbb{R}^{n+1}_+)$. According to Theorem 2.1.8 and since $\omega \in S'(\mathbb{R}^n) \cap L_1^{\mathrm{loc}}(\mathbb{R}^n)$ it follows for $t > 0$ and $x \in \mathbb{R}^n$

$$(-\Delta_x)^\alpha W_t^\alpha \omega(x) = [(-\Delta_x)^\alpha G_t^\alpha] * \omega(x) = -[\partial_t G_t^\alpha] * \omega(x) = -\partial_t W_t^\alpha \omega.$$

The last equality follows from Lebesgue's dominated convergence theorem.
Step 2. By Proposition 2.3.2 we have for F

$$(\partial_t + (-\Delta_x)^\alpha)(G^\alpha * F) = F \quad \text{in } D'(\mathbb{R}^{n+1}).$$

Because of $F|_{\mathbb{R}^{n+1}_+} = f$ and $\operatorname{supp} F \subset \mathbb{R}^{n+1}_+$ this yields

$$(\partial_t + (-\Delta_x)^\alpha)(G^\alpha * F) = f \quad \text{in } D'(\mathbb{R}^{n+1}_+) \tag{2.3.12}$$

and together with (2.3.10)

$$\big(\partial_t + (-\Delta_x)^\alpha\big)(W^\alpha_t \omega + G^\alpha * F) = f \quad \text{in } D'(\mathbb{R}^{n+1}_+). \tag{2.3.13}$$

Under the above assumptions we have by means of Remark 2.1.5 $H = G^\alpha * F \in D'(\mathbb{R}^{n+1})$ and $h = G^\alpha * f \in D'(\mathbb{R}^{n+1}_+)$. Thus, the assertions (2.3.8) and (2.3.9) follow. □

We conclude the section with a lemma that states under which conditions the solution spaces $L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n))$ as given in Definition 1.2.7 consist entirely of regular distributions.

Lemma 2.3.5. Let $1 \le p, q \le \infty$ and $s > 0$ ($p < \infty$ for F-spaces). Let $T > 0$, $1 < v \le \infty$ and $b < 1 - \frac{1}{v}$. Let

$$U(x,t) = \begin{cases} u(x,t), & x \in \mathbb{R}^n,\ t \in (0,T), \\ 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R} \setminus (0,T) \end{cases}$$

where $u \in L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n))$. Then $U \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$.

Proof. Under the conditions on p, q, s any $u(\cdot,t) \in A^s_{p,q}(\mathbb{R}^n)$ belongs for fixed t to $L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and to $L_p(\mathbb{R}^n)$, cf. Theorem 1.4.1 and Corollary 1.4.2. Thus, there exists for any compact subset $K \subset \mathbb{R}^{n+1}$ some $R > 0$ such that

$$\int_K |U(x,t)|\,\mathrm{d}(x,t) \le \int_0^T \int_{|x|\le R} |u(x,t)|\mathrm{d}x\,\mathrm{d}t$$

$$\le c \int_0^T \|u(\cdot,t)|L_p(\mathbb{R}^n)\|\mathrm{d}t \le c\int_0^T t^{-b}t^b \|u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\mathrm{d}t \tag{2.3.14}$$

$$\le c\left(\int_0^T t^{-bv'}\mathrm{d}t\right)^{1/v'} \|u|L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n))\| \tag{2.3.15}$$

$$\le c\|u|L_v((0,T),\, b,\, A^s_{p,q}(\mathbb{R}^n))\|, \quad \text{if } b < 1 - \frac{1}{v}.$$

The first part in (2.3.14) and (2.3.15) are obtained by means of Hölder's inequality. □

2.4 Cauchy's problem for the generalized Navier-Stokes equations

In Chapter 5 we deal with Cauchy's problem for the generalized Navier-Stokes equations. We assume $n \in \mathbb{N}$ with $n \ge 2$, $\alpha \in \mathbb{N}$ and $0 < T \le \infty$. As in the case of the generalized nonlinear heat equation the diffusion part $\partial_t - \Delta_x$ is replaced by $\partial_t + (-\Delta_x)^\alpha$. Thus, the Cauchy problem for the generalized Navier-Stokes equations are given by

$$\begin{aligned} \partial_t \mathbf{u} + (-\Delta_x)^\alpha \mathbf{u} + (\mathbf{u},\nabla)\mathbf{u} + \nabla P &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{2.4.1}$$

Hence, the case $\alpha = 1$ corresponds to the classical Navier-Stokes equations. Recall that

$$[(\mathbf{u},\nabla)\mathbf{u}]^k = \sum_{j=1}^n u^j \partial_j u^k, \quad k = 1,\dots,n.$$

As usual the divergence of a vector-valued function $\mathbf{u}$ and the gradient of a scalar function P are given by

$$\operatorname{div} \mathbf{u} = \sum_{j=1}^n \partial_j u^j, \qquad \text{and } \nabla P = (\partial_1 P, \dots, \partial_n P).$$

Since $\operatorname{div} \mathbf{u} = 0$ it holds that

$$(\mathbf{u},\nabla)\mathbf{u} = \operatorname{div}(\mathbf{u} \otimes \mathbf{u}),$$

where $\mathbf{u} \otimes \mathbf{u}$ is the usual tensor product such that the k-th component of $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$ is given by

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{u})^k = \sum_{j=1}^n \partial_j (u^j u^k).$$

Commonly one incorporates the request $\operatorname{div} \boldsymbol{u} = 0$ by means of the so-called Leray projector $\mathbb{P}$ given in terms of scalar Riesz transforms R_k as described below. This leads to the following reformulation of (2.4.1)

$$\begin{aligned} \partial_t \mathbf{u} + (-\Delta_x)^\alpha \mathbf{u} + \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0 \quad \text{in } \mathbb{R}^n \end{aligned} \tag{2.4.2}$$

which we will examine in detail in Chapter 5. The next part recalls necessary facts about Riesz transform and Leray projector following the presentation in [37, Sections 1.2.6 and 6.1.2]

2.4.1 Riesz transform and Leray projector

Let $L_2(\mathbb{R}^n)_n$ be the Hilbert space of all complex-valued functions $\boldsymbol{f} = (f_1,\dots,f_n)$ with $f_k \in L_2(\mathbb{R}^n)$, $k = 1,\dots,n$ normed by

$$\|\boldsymbol{f} \; |L_2(\mathbb{R}^n)_n\| = \left(\sum_{k=1}^n \|f_k \; |L_2(\mathbb{R}^n)\|^2 \right)^{1/2}.$$

Further, let

$$\operatorname{div} L_2(\mathbb{R}^n)_n = \{f \in L_2(\mathbb{R}^n)_n : \operatorname{div} \boldsymbol{f} = 0\}$$

be the closed subspace of $L_2(\mathbb{R}^n)_n$ containing all divergence-free functions in $L_2(\mathbb{R}^n)_n$ where the derivatives must be interpreted in the framework of $S'(\mathbb{R}^n)$. We are interested in an orthogonal projection from $L_2(\mathbb{R}^n)_n$ onto $\operatorname{div} L_2(\mathbb{R}^n)_n$. Let R_k,

$$R_k f(\cdot) = i \left(\frac{\xi_k}{|\xi|} \hat{f} \right)^{\vee} (\cdot), \quad k = 1, \ldots, n, \tag{2.4.3}$$

be the usual Riesz transform, which can be expressed in terms of singular integrals

$$R_k f(x) = c_n \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{y_k}{|y|^{n+1}} f(x-y) \mathrm{d}y, \quad k = 1, \ldots, n, \tag{2.4.4}$$

$x \in \mathbb{R}^n$. For convenience we introduce the operator $\mathbb{Q}$ given on the Fourier side by

$$(\widehat{\mathbb{Q} \boldsymbol{f}}\,)^k(\xi) = \frac{\xi_k}{|\xi|^2} \sum_{j=1}^n \xi_j \widehat{f^j} = -i \frac{\xi_k}{|\xi|^2} (\operatorname{div} \boldsymbol{f}\,)^\wedge(\xi), \quad k = 1, \ldots, n. \tag{2.4.5}$$

It can be written in terms of the Riesz transform

$$(\widehat{\mathbb{Q} \boldsymbol{f}}\,)^k(\xi) = \frac{\xi_k}{|\xi|} \sum_{j=1}^n \frac{\xi_j \widehat{f^j}}{|\xi|} = -R_k \sum_{j=1}^n R_j f^j \quad k = 1, \ldots, n.$$

Then the Leray projector $\mathbb{P}$ is given by

$$(\mathbb{P}\, \mathbf{f})^k = f^k + R_k \sum_{j=1}^n R_j f^j \quad k = 1, \ldots, n \tag{2.4.6}$$

and hence, can be represented by $\mathbb{P} = \mathrm{id} - \mathbb{Q}$.

Proposition 2.4.1. *Let $L_2(\mathbb{R}^n)_n$, $\operatorname{div} L_2(\mathbb{R}^n)_n$ and $\mathbb{P}$ be defined as above. Then $\mathbb{P}$ is the orthogonal projection of $L_2(\mathbb{R}^n)_n$ onto $\operatorname{div} L_2(\mathbb{R}^n)_n$. More precisely it holds*

$$\mathbb{P}^2 = \mathbb{P} = \mathbb{P}^* \tag{2.4.7}$$

and

$$\mathbb{P}\, L_2(\mathbb{R}^n)_n = \operatorname{div} L_2(\mathbb{R}^n)_n. \tag{2.4.8}$$

Furthermore,

$$\mathbb{P}\, \nabla f = 0 \quad \textit{for any } f \in L_2(\mathbb{R}^n)_n. \tag{2.4.9}$$

Proof. For a proof we refer to [37, Proposition 6.1]. □

For later use we recall that $R_k : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$, $1 < p < \infty$ is a linear and bounded mapping. Further, the following assertion holds.

Proposition 2.4.2. *Let* $1 < p < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$R_k : \quad A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(\mathbb{R}^n) \tag{2.4.10}$$

where A *stands either for* F *or* B.

Proof. For a proof we refer to [37, Theorem 1.25] and [38, Theorem 3.52]. □

3 A generalized nonlinear heat equation

In this chapter we deal with the generalized nonlinear heat equation

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{3.0.1}$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $Du^2 = \sum_{j=1}^{n} \partial_j u^2$. The case $\alpha = 1$ corresponds to a classical nonlinear heat equation. In our approach, a solution of (3.0.1) is considered as fixed point of the operator T_{u_0} given as

$$T_{u_0} u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T \tag{3.0.2}$$

in some appropriate function spaces with W_t^α as given in Definition 2.3.3. As we have seen in Proposition 2.2.1, $G^\alpha(x,t)$ is a fundamental solution of the generalized heat equation (2.2.1). There we also introduced the notation $G_t^\alpha(x) = G^\alpha(x,t)$ if we fix $t > 0$.

We are interested in solutions u of (3.0.1) belonging to some Besov or Triebel-Lizorkin space $A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$, with respect to the space variable. Because of the structure of the nonlinearity these spaces have to fulfill certain multiplication properties. During this chapter we assume that they are multiplication algebras which holds if $s > \frac{n}{p}$ and in some limiting cases $s = \frac{n}{p}$, cf. Theorem 1.4.3. Concerning the initial data, we suppose $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$ with $s - \alpha < s_0 \leq s$. The time dependency of the solution is described in terms of some weighted Lebesgue spaces as introduced in Definition 1.2.7.

Our method is as follows. We solve the fixed point problem (3.0.2) for the operator T_{u_0} in our solution space. For this purpose we need a-priori estimates in some spaces $A^s_{p,q}(\mathbb{R}^n)$ for both terms of the right hand side of (3.0.2). The key estimate, formulated in Theorem 3.2.2, reads as

$$t^{d/2\alpha} \| W_t^\alpha \omega | A^{s+d}_{p,q}(\mathbb{R}^n) \| \leq c \| \omega | A^s_{p,q}(\mathbb{R}^n) \|, \quad 0 < t \leq 1 \tag{3.0.3}$$

if $1 \leq p, q \leq \infty$ ($p < \infty$ in case of F- spaces), $s \in \mathbb{R}$, $d \geq 0$, and $\alpha \in \mathbb{N}$. To achieve (3.0.3), we apply decomposition methods by means of wavelets and molecules to these spaces. More precisely, having an appropriate representation of $\omega \in A^s_{p,q}(\mathbb{R}^n)$ by means

of sufficiently smooth Daubechies wavelets, we show that $W_t^\alpha \omega$ can be represented by means of molecules in $A^{s+d}_{p,q}(\mathbb{R}^n)$.
The main result is contained in Theorem 3.3.5, where existence and uniqueness of a local mild solution of (3.0.1) in the sense of (3.0.2) is proved for arbitrary initial values $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$ with $s - \alpha < s_0 \leq s$. Moreover we show that the solution is strong. Detailed explanations what we mean by mild and strong solutions follow in Section 3.3. Finally, we conclude the chapter with assertions on stability and well-posedness.

3.1 α- caloric wavelets

From now on we assume that $\omega \in A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$. Under the conditions of Proposition 1.3.3, we can represent

$$\omega = \sum_{j,G,m} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in a^s_{p,q}(\mathbb{R}^n) \tag{3.1.1}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(\omega) = 2^{jn/2} \left\langle \omega, \Psi^j_{G,m} \right\rangle . \tag{3.1.2}$$

We are interested in a similar decomposition of $W_t^\alpha \omega$. We split

$$\omega = \omega^0 + \omega^u, \quad u \in \mathbb{N}, \tag{3.1.3}$$

where

$$\omega^0 = \sum_{m \in \mathbb{Z}^n} \lambda_m \Psi_m, \quad \lambda_m = \lambda_m(\omega) = \left\langle \omega, \Psi_m \right\rangle , \tag{3.1.4}$$

with

$$\Psi_m(x) = \prod_{l=1}^n \psi_F(x_l - m_l), \quad m \in \mathbb{Z}^n \tag{3.1.5}$$

and

$$\omega^u = \sum_{j=0}^\infty \sum_{G \in G^*} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \tag{3.1.6}$$

with (3.1.2) and $G^* = \{F, M\}^{n*}$, i.e., at least one of the G_l, $l = 1, \dots, n$ is an M. It will be essential that $u \in \mathbb{N}$ in (1.3.1) can be chosen arbitrarily large. That is the reason why we indicated the chosen $u \in \mathbb{N}$ in (3.1.6). Moreover, any $\Psi^j_{G,m}$ in (3.1.6) satisfies moment conditions up to u as in (1.3.8). With $W_t^\alpha \omega$ as in Definition 2.3.3 we obtain for $t > 0$

$$\begin{aligned} W_t^\alpha \omega(x) &= W_t^\alpha \omega^0(x) + W_t^\alpha \omega^u(x) \\ &= \sum_{m\in\mathbb{Z}^n} \lambda_m W_t^\alpha \Psi_m(x) + \sum_{j=0}^{\infty} \sum_{G\in G^*} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2} W_t^\alpha \Psi_{G,m}^j(x). \qquad (3.1.7) \end{aligned}$$

As we will see below, (3.1.7) is for each fixed $t > 0$ a representation of $W_t^\alpha \omega$ by molecules belonging to $A_{p,q}^{s+d}(\mathbb{R}^n)$ with some $d \geq 0$.
Thus the unconditional convergence of the series is preserved at least as limit in $S'(\mathbb{R}^n)$ as a dual pairing with respect to $\left(e^{-t|\xi|^{2\alpha}}\right)^\vee \in S(\mathbb{R}^n)$. Let

$$\begin{aligned} b_{G,m}^j(x,t) &:= 2^{-jn/2} W_t^\alpha \Psi_{G,m}^j(x) = \int_{\mathbb{R}^n} G_t^\alpha(x-y) 2^{-jn/2} \Psi_{G,m}^j(y) \mathrm{d}y \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee (x-y) \prod_{l=1}^n \psi_{G_l}(2^j y_l - m_l) \mathrm{d}y. \qquad (3.1.8) \end{aligned}$$

According to the case $\alpha = 1$, cf. [37, Subsection 2.4.2], the functions $b_{G,m}^j(x,t)$ are called α-caloric wavelets. As already mentioned, we show that after a slight modification they are molecules in the sense of Definition 1.3.6 for appropriately chosen parameters N, K, L.

Proposition 3.1.1. *Let $\alpha \in \mathbb{N}$ and let $b_{G,m}^j(x,t)$ have the meaning of (3.1.8) based on (1.3.6), where ψ_M, $\psi_F \in C^u(\mathbb{R})$ are Daubechies wavelets with a given $u \in \mathbb{N}$. We put*

$$b_{G,m}^j(x,t)_d = C\, 2^{jd}\, t^{d/2\alpha}\, b_{G,m}^j(x,t), \quad j \in \mathbb{N}_0,\ G \in G^*,\ m \in \mathbb{Z}^n. \qquad (3.1.9)$$

Then there exists $C > 0$ such that the functions $b_{G,m}^j(x,t)_d$ are (N,K,L)- molecules according to Definition 1.3.6 for any fixed t with $2^j t^{1/2\alpha} \geq 1$, provided that $N \leq u$, $K \leq u$, $N + n - 1 < L < u + n - d$ and $0 \leq d < u - N + 1$.

Proof. **Step 1.** We start with the moment conditions. It is sufficient to consider $b_{G,m}^j(x,t)$. Let $|\beta| < u$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} x^\beta b_{G,m}^j(x,t) \mathrm{d}x &= \int_{\mathbb{R}^n} x^\beta \left(\int_{\mathbb{R}^n} G_t^\alpha(x-y) \prod_{l=1}^n \psi_{G_l}(2^j y_l - m_l) \mathrm{d}y \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G_t^\alpha(z)(y+z)^\beta \mathrm{d}z \right) \prod_{l=1}^n \psi_{G_l}(2^j y_l - m_l) \mathrm{d}y \\ &= \int_{\mathbb{R}^n} G_t^\alpha(z) \left(\prod_{l=1}^n \int_{\mathbb{R}} (y_l + z_l)^{\beta_l} \psi_{G_l}(2^j y_l - m_l) \mathrm{d}y_l \right) \mathrm{d}z \\ &= 0, \end{aligned}$$

according to (1.3.8), since at least one of the $G_l = M$. So the moment conditions (1.3.26) hold for $b^j_{G,m}(\cdot, t)_d$ if $N \leq u$.
Step 2. We prove (1.3.25) with $|\gamma| = 0$ where we may assume $m = 0$, i.e. we wish to estimate

$$b^j_G(x,t) = b^j_{G,0}(x,t) = \int\limits_{\mathbb{R}^n} G^\alpha_t(x-y) \prod_{l=1}^n \psi_{G_l}(2^j y_l)\mathrm{d}y \tag{3.1.10}$$

where $j \in \mathbb{N}_0$, $G \in G^*$, $2^j t^{1/2\alpha} \geq 1$. We rewrite

$$b^j_G(x,t) = \int\limits_{\mathbb{R}^n} t^{-n/2\alpha} G^\alpha \left(\frac{x-y}{t^{1/2\alpha}}\right) \prod_{l=1}^n \psi_{G_l}(2^j y_l)\mathrm{d}y \tag{3.1.11}$$

where $G^\alpha(x) = (2\pi)^{-n/2}\left(e^{-|\eta|^{2\alpha}}\right)^\vee(x)$. Apparently,

$$b^j_G(t^{1/2\alpha}\, x, t) = \int\limits_{\mathbb{R}^n} G^\alpha(x-y) \prod_{l=1}^n \psi_{G_l}(2^j\, t^{1/2\alpha}\, y_l)\mathrm{d}y. \tag{3.1.12}$$

Next we expand G^α at the origin in a Taylor polynomial with remainder term of order u and substitute it into (3.1.12). Because of the moment conditions of $\Psi^j_{G,m}$ terms of order less than u vanish and we only have to estimate

$$\begin{aligned} \left|b^j_G(t^{1/2\alpha}\, x, t)\right| &\lesssim \int\limits_{\mathbb{R}^n} \sum_{|\beta|=u} \left|(D^\beta G^\alpha)(x-\xi)\, y^\beta \prod_{l=1}^n \psi_{G_l}(2^j\, t^{1/2\alpha}\, y_l)\right|\mathrm{d}y \\ &\lesssim \int\limits_{\mathbb{R}^n} \sum_{|\beta|=u} \left|y^\beta\right| \left|\prod_{l=1}^n \psi_{G_l}(2^j\, t^{1/2\alpha}\, y_l)\right| \mathrm{d}y \\ &\lesssim \int\limits_{\mathbb{R}^n} |y|^u \left|\prod_{l=1}^n \psi_{G_l}(2^j\, t^{1/2\alpha} y_l)\right| \mathrm{d}y \end{aligned} \tag{3.1.13}$$

where we used the boundedness of the derivatives of G^α in (3.1.13). Since the integrand is zero outside a ball of radius $c\, 2^{-j}\, t^{-1/2\alpha}$ centered at the origin, we obtain

$$|b^j_G(t^{1/2\alpha}\, x, t)| \lesssim \int\limits_{|y|<c\, 2^{-j} t^{-1/2\alpha}} |y|^u \mathrm{d}y \leq c\, \left(2^{-j} t^{-1/2\alpha}\right)^{u+n}, \quad \forall x \in \mathbb{R}^n. \tag{3.1.14}$$

On the other hand, it follows from (3.1.12) and $G^\alpha \in S(\mathbb{R}^n)$

$$
\begin{aligned}
|b_G^j(t^{1/2\alpha}x,t)| &\lesssim \int\limits_{|y|<c\,2^{-j}t^{-1/2\alpha}} \left|\left(e^{-|\eta|^{2\alpha}}\right)^\vee (x-y)\right| \mathrm{d}y \\
&\leq \int\limits_{|y|<c\,2^{-j}t^{-1/2\alpha}} \frac{c_L}{1+|x-y|^L} \mathrm{d}y \\
&\lesssim \frac{c_L}{(1+|x|)^L} \int\limits_{|y|<c\,2^{-j}t^{-1/2\alpha}} (1+|y|)^L \mathrm{d}y \\
&\lesssim \frac{c_L}{(1+|x|)^L} \int\limits_{|y|<c\,2^{-j}t^{-1/2\alpha}} \mathrm{d}y \\
&\sim \frac{c_L}{(1+|x|)^L} (2^{-j}t^{-1/2\alpha})^n
\end{aligned} \tag{3.1.15}
$$

for all $L > 0$ and $2^j t^{1/2\alpha} \geq 1$. Let now $0 < \varepsilon < 1$. Using both estimates, (3.1.14) and (3.1.15), we find

$$
\begin{aligned}
|b_G^j(t^{1/2\alpha}x,t)| &= |b_G^j(t^{1/2\alpha}x,t)|^\varepsilon |b_G^j(t^{1/2\alpha}x,t)|^{(1-\varepsilon)} \\
&\leq \frac{c_{\varepsilon,L}}{(1+|x|)^L} \left(2^{-j}t^{-1/2\alpha}\right)^{\varepsilon n} \left(2^{-j}t^{-1/2\alpha}\right)^{(1-\varepsilon)(u+n)} \qquad (3.1.16) \\
&= \frac{c_{\varepsilon,L}}{(1+|x|)^L} \frac{\left(2^{-j}t^{-1/2\alpha}\right)^{u-\varepsilon u+n-L}}{\left(2^{-j}t^{-1/2\alpha}\right)^{-L}} \\
&\leq \frac{c_{\varepsilon,L}}{(1+2^j t^{1/2\alpha}|x|)^L} \left(2^{-j}t^{-1/2\alpha}\right)^{u-\varepsilon u+n-L}
\end{aligned}
$$

since $2^j t^{1/2\alpha} \geq 1$. Replacing $t^{1/2\alpha}x \mapsto x$ in (3.1.16) yields

$$
|b_G^j(x,t)| \leq \frac{c_{\varepsilon,L}}{(1+2^j|x|)^L} \left(2^{-j}t^{-1/2\alpha}\right)^{u-\varepsilon u+n-L}. \tag{3.1.17}
$$

Let now $0 \leq d < u - N + 1$. We choose $L > 0$ and $0 < \varepsilon < 1$ such that

$$
N + n - 1 < L \leq (1-\varepsilon)u + n - d.
$$

Thus, it follows

$$
\left(2^{-j}t^{-1/2\alpha}\right)^{(1-\varepsilon)u+n-L} \leq \left(2^{-j}t^{-1/2\alpha}\right)^d. \tag{3.1.18}
$$

Hence, we have shown

$$
|b_G^j(x,t)| \leq \frac{c_{\varepsilon,L}}{(1+2^j|x|)^L} \left(2^{-j}t^{-1/2\alpha}\right)^d
$$

for any $d \geq 0$ with $d < u - N + 1$.
Step 3. Let $|\gamma| \geq 1$. Then,

$$|D_x^\gamma b_G^j(t^{1/2\alpha}\,x,t)| \;\leq\; 2^{|\gamma|j} t^{|\gamma|/2\alpha} \int_{\mathbb{R}^n} |G^\alpha(x-y)(D^\gamma \prod_{l=1}^{n} \psi_{G_l})(2^j t^{|\gamma|/2\alpha} y_l)| \mathrm{d}y$$

in the classical sense if $|\gamma| \leq K \leq u$. Because of the compactness of supp ψ_{G_l}, the moment conditions also hold for the derivatives of ψ_M up to order u, which is seen by integration by parts. Hence, we obtain

$$|D_x^\gamma b_{G,m}^j(t^{1/2\alpha}\,x,t)| = t^{|\gamma|/2\alpha}|(D^\gamma b_{G,m}^j)(t^{1/2\alpha}\,x,t)| \lesssim 2^{j|\gamma|} t^{|\gamma|/2\alpha} \left(2^{-j} t^{-1/2\alpha}\right)^{u+n}. \tag{3.1.19}$$

A similar calculation as in Step 2 yields

$$|D_x^\gamma b_{G,m}^j(x,t)| \leq c\, 2^{-jd} t^{-d/2\alpha} \frac{2^{j|\gamma|}}{(1+2^j|x|)^L} \tag{3.1.20}$$

with N, K, L as above. Thus, the functions in (3.1.9) are (N,K,L) -molecules for $2^j t^{1/2\alpha} \geq 1$. □

Theorem 3.1.2. *Let $0 < p,q \leq \infty$ ($p < \infty$ for the F-spaces), $s \in \mathbb{R}$. Let $d \geq 0$ such that $s+d \geq 0$, $\alpha \in \mathbb{N}$, and $u \in \mathbb{N}$ with*

$$u > d + \begin{cases} \max(s,\sigma_p), & \textit{for } B\textit{-spaces} \\ \max(s,\sigma_{p,q}), & \textit{for } F\textit{-spaces.} \end{cases} \tag{3.1.21}$$

Then, the numbers N, K, L in Proposition 3.1.1 can be chosen such that for some $C > 0$ and any t with $2^j t^{1/2\alpha} \geq 1$

$$b_{G,m}^j(x,t)_d = C\, 2^{jd}\, t^{d/2\alpha}\, b_{G,m}^j(x,t), \quad j \in \mathbb{N}_0,\ G \in G^*,\ m \in \mathbb{Z}^n$$

are molecules for $A_{p,q}^{s+d}(\mathbb{R}^n)$ in the sense of Proposition 1.3.8.

Proof. We restrict ourselves to the B-spaces. First we choose $N := [\sigma_p] + 1$. Then $N > \sigma_p - (s+d)$. Since $u > \sigma_p = [\sigma_p] + \{\sigma_p\}$ it follows that $u \geq N$. So the moment conditions hold for $W_t^\alpha \Psi_{G,m}^j$ with the above chosen N. Let now $n + \sigma_p < L < u + n - d$. Because of the choice of N we have $L > N + n - 1$, too. From $u > d + \sigma_p$ it follows that $d < u - \sigma_p < u - N + 1$. Regarding the derivatives of $b_{G,m}^j(x,t)_d$, we claim $s + d < K \leq u$. Hence, the conditions of both, Proposition 3.1.1 and Proposition 1.3.8 are satisfied. Replacing σ_p by $\sigma_{p,q}$ leads to similar results for the F-spaces. □

Later on we only need spaces $A_{p,q}^s(\mathbb{R}^n)$ with $p,q \geq 1$. Hence, the considered spaces are Banach spaces. This reduces the assumptions with respect to u, N, K and L as follows.

Corollary 3.1.3. *Let $1 \le p, q \le \infty$ ($p < \infty$ for the F-spaces), $s \in \mathbb{R}$. Let $d \ge 0$ such that $s + d \ge 0$, $\alpha \in \mathbb{N}$ and $u \in \mathbb{N}$ with*

$$u > d + \max(s, 0). \tag{3.1.22}$$

Then there exists $C > 0$ such that

$$b^j_{G,m}(x,t)_d = C\, 2^{jd}\, t^{d/2\alpha}\, b^j_{G,m}(x,t), \quad j \in \mathbb{N}_0,\ G \in G^*,\ m \in \mathbb{Z}^n$$

for any t with $2^j t^{1/2\alpha} \ge 1$ are molecules for $A^{s+d}_{p,q}(\mathbb{R}^n)$ in the sense of Proposition 1.3.8 if $N = 1$, $K = u$, and $n < L < u + n - d$.

3.2 Mapping properties related to the heat equation

In order to prove the above mentioned a-priori estimate (3.0.3), we first derive a corresponding result for ω^u defined in (3.1.6).

Proposition 3.2.1. *Let $1 \le p, q \le \infty$ ($p < \infty$ for the F-spaces), $s \in \mathbb{R}$. Let $d \ge 0$ such that $s + d \ge 0$, $\alpha \in \mathbb{N}$, and $u \in \mathbb{N}$ with*

$$u > d + \max(s, 0) \tag{3.2.1}$$

Then there exists a constant $c > 0$ such that for any $t \ge 1$ and all $\omega \in A^s_{p,q}(\mathbb{R}^n)$

$$t^{d/2\alpha} \| W^\alpha_t \omega^u | A^{s+d}_{p,q}(\mathbb{R}^n) \| \le c \| \omega | A^s_{p,q}(\mathbb{R}^n) \|, \tag{3.2.2}$$

whereas ω^u is given by (3.1.6).

Proof. $t \ge 1$ ensures $2^j t^{1/2\alpha} \ge 1$ for $j \in \mathbb{N}_0$. Consequently, we can apply Corollary 3.1.3. Taking into account the molecular representation of $W^\alpha_t \omega^u$, we write

$$\begin{aligned} W^\alpha_t \omega^u &= \sum_{j=0}^{\infty} \sum_{G \in G^*} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m 2^{-jn/2} W^\alpha_t \Psi^j_{G,m} \\ &= \sum_{j=0}^{\infty} \sum_{G \in G^*} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m\, b^j_{G,m}(\cdot, t) \\ &= \sum_{j=0}^{\infty} \sum_{G \in G^*} \sum_{m \in \mathbb{Z}^n} \mu^{j,G}_m\, b^j_{G,m}(\cdot, t)_d \end{aligned} \tag{3.2.3}$$

with

$$C\, \mu^{j,G}_m = 2^{-jd} t^{-d/2\alpha} \lambda^{j,G}_m \tag{3.2.4}$$

where $C > 0$ has the same meaning as in (3.1.9). Let

$$\mu^* = \{\mu_m^{j,G} : \ j \in \mathbb{N}_0, \ G \in G^*, \ m \in \mathbb{Z}^n\} \tag{3.2.5}$$

and similarly λ^*. We restrict ourselves to the B-spaces. The proof for the F-spaces is similar. By Proposition 1.3.8, it remains to show that $\mu^* \in b_{p,q}^{s+d}(\mathbb{R}^n)$. It holds

$$\begin{aligned}
\|\mu^* | b_{p,q}^{s+d}(\mathbb{R}^n)\| &= \left(\sum_{j=0}^{\infty} 2^{j(s+d-\frac{n}{p})q} \sum_{G \in G^*} \left(\sum_{m \in \mathbb{Z}^n} |\mu_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\sim \left(\sum_{j=0}^{\infty} 2^{jdq} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^*} \left(\sum_{m \in \mathbb{Z}^n} 2^{-jdp} t^{-dp/2\alpha} |\lambda_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&= t^{-d/2\alpha} \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^*} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&= t^{-d/2\alpha} \|\lambda^* | b_{p,q}^s(\mathbb{R}^n)\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|W_t^\alpha \omega^u | B_{p,q}^{s+d}(\mathbb{R}^n)\| &\leq c \, \|\mu^* | b_{p,q}^{s+d}(\mathbb{R}^n)\| \sim t^{-d/2\alpha} \|\lambda^* | b_{p,q}^s(\mathbb{R}^n)\| \\
&\leq c \, t^{-d/2\alpha} \|\lambda \, | b_{p,q}^s(\mathbb{R}^n)\| \sim t^{-d/2\alpha} \|\omega | B_{p,q}^s(\mathbb{R}^n)\|,
\end{aligned} \tag{3.2.6}$$

where we used Proposition 1.3.3. □

Theorem 3.2.2. *Let $1 \leq p, q \leq \infty$ ($p < \infty$ for the F-spaces), $s \in \mathbb{R}$, $d \geq 0$ and $\alpha \in \mathbb{N}$. Then there is a constant $c > 0$ such that*

$$t^{d/2\alpha} \|W_t^\alpha \omega | A_{p,q}^{s+d}(\mathbb{R}^n)\| \leq c \|\omega | A_{p,q}^s(\mathbb{R}^n)\| \tag{3.2.7}$$

for all t with $0 < t \leq 1$ and $\omega \in A_{p,q}^s(\mathbb{R}^n)$.

Proof. **Step 1.** We prove (3.2.7) for $s + d \geq 0$ and take advantage of the representation of $\omega \in A_{p,q}^s(\mathbb{R}^n)$ as in Proposition 1.3.3. Let $2^{-2\alpha k} < t \leq 2^{-2\alpha(k-1)}$, $k \in \mathbb{N}$. Instead of (3.1.3)-(3.1.6) we split ω into

$$\omega = \omega_k^0 + \omega_k = \sum_{j<k} \sum_{G,m} \lambda_m^{j,G} 2^{-jn/2} \Psi_{G,m}^j + \sum_{j \geq k} \sum_{G,m} \lambda_m^{j,G} 2^{-jn/2} \Psi_{G,m}^j. \tag{3.2.8}$$

We do not indicate $u \in \mathbb{N}$ chosen according to 3.1.22. If $j \geq k$ then $2^j t^{1/2\alpha} > 2^j 2^{-k} = 2^{j-k} \geq 1$. Applying Corollary 3.1.3 and Proposition 1.3.8, we find as in Proposition 3.2.1

$$t^{d/2\alpha} \, \|W_t^\alpha \omega_k | A_{p,q}^{s+d}(\mathbb{R}^n)\| \leq c \, \|\omega | A_{p,q}^s(\mathbb{R}^n)\|, \tag{3.2.9}$$

where $c>0$ is independent of $k\in\mathbb{N}$, $2^{-2\alpha k}<t\le 2^{-2\alpha(k-1)}$ and $\omega\in A^s_{p,q}(\mathbb{R}^n)$.
Let now $j<k$ and $A=B$. Then,

$$\begin{aligned}\|\omega_k^0|B^{s+d}_{p,q}(\mathbb{R}^n)\| &\sim \left(\sum_{j<k}2^{j(s+d-\frac{n}{p})q}\sum_{G\in G_j}\left(\sum_{m\in\mathbb{Z}^n}|\lambda^{j,G}_m|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\ &\le 2^{kd}\left(\sum_{j<k}2^{j(s-\frac{n}{p})q}\sum_{G\in G_j}\left(\sum_{m\in\mathbb{Z}^n}|\lambda^{j,G}_m|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\ &\lesssim 2^{kd}\|\lambda\,|b^s_{p,q}(\mathbb{R}^n)\|. \end{aligned}\tag{3.2.10}$$

Similarly for the F-spaces. By means of the generalized Minkowski's inequality and (3.2.10) we obtain

$$\|W_t^\alpha\omega_k^0|A^{s+d}_{p,q}(\mathbb{R}^n)\| \le \int_{\mathbb{R}^n}G_t^\alpha(y)\|\omega_k^0(x-y)|A^{s+d}_{p,q}(\mathbb{R}^n)\|\mathrm{d}y \tag{3.2.11}$$

$$\le \|\omega_k^0|A^{s+d}_{p,q}(\mathbb{R}^n)\|\int_{\mathbb{R}^n}G_t^\alpha(y)\mathrm{d}y\le c\,2^{kd}\|\lambda|a^s_{p,q}(\mathbb{R}^n)\| \tag{3.2.12}$$

$$\sim c\,2^{kd}\|\omega|A^s_{p,q}(\mathbb{R}^n)\|. \tag{3.2.13}$$

Because of the assumption $2^{-2\alpha k}<t\le 2^{-2\alpha(k-1)}$, it follows that $t^{-d/2\alpha}\sim 2^{kd}$. Consequently,

$$t^{d/2\alpha}\,\|W_t^\alpha\omega_k^0|A^{s+d}_{p,q}(\mathbb{R}^n)\|\le c\,\|\omega|A^s_{p,q}(\mathbb{R}^n)\|. \tag{3.2.14}$$

Together with (3.2.9) the assertion follows for $s+d\ge 0$.
Step 2. It remains to show the case $s+d<0$. For this purpose consider the lift operator I_δ, $\delta\in\mathbb{R}$ introduced in (1.2.11). Using (2.3.3) we see that

$$\begin{aligned}\mathcal{F}(W_t^\alpha(I_\delta\omega))(\xi) &= \mathcal{F}(W_t^\alpha(\mathcal{F}^{-1}[(1+|\cdot|^2)^{-\frac{\delta}{2}}\mathcal{F}\omega])(\xi)\\ &= e^{-t|\xi|^{2\alpha}}(1+|\xi|^2)^{-\frac{\delta}{2}}\mathcal{F}\omega(\xi)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(I_\delta(W_t^\alpha\omega))(\xi) &= (1+|\xi|^2)^{-\frac{\delta}{2}}\mathcal{F}(W_t^\alpha\omega)(\xi)\\ &= e^{-t|\xi|^{2\alpha}}(1+|\xi|^2)^{-\frac{\delta}{2}}\mathcal{F}\omega(\xi).\end{aligned}$$

Thus, $W_t^\alpha=I_{-\delta}W_t^\alpha I_\delta$. Let $s+d<0$ and $\delta\in\mathbb{R}$ such that $s+d-\delta\ge 0$. Then we have by (1.2.12)

$$\begin{aligned}&t^{d/2\alpha}\|W_t^\alpha\omega|A^{s+d}_{p,q}(\mathbb{R}^n)\|=t^{d/2\alpha}\|I_{-\delta}W_t^\alpha I_\delta\omega|A^{s+d}_{p,q}(\mathbb{R}^n)\|\\ \sim\ &t^{d/2\alpha}\|W_t^\alpha I_\delta\omega|A^{s+d-\delta}_{p,q}\|\le c\,\|I_\delta\omega|A^{s-\delta}_{p,q}\|\\ \sim\ &c\,\|\omega|A^s_{p,q}(\mathbb{R}^n)\|\end{aligned}\tag{3.2.15}$$

which completes the proof. $\square$

For later purposes we consider a similar estimate for

$$W(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha f(\cdot,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0<t<T \tag{3.2.16}$$

with f in an appropriately chosen function space.

Proposition 3.2.3. *Let $1 \le p,q \le \infty$ ($p<\infty$ for the F-spaces), $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}$. Let $T>0$ and*

$$\frac{1}{\alpha} < v \le \infty,\ -\infty < a < \alpha - \frac{1}{v},\ 0 \le d < 2\left(\alpha - \frac{1}{v}\right),\ -\infty < \alpha g \le d. \tag{3.2.17}$$

Let

$$u_0 \in A_{p,q}^{s+\alpha g}(\mathbb{R}^n) \quad \textit{and } f \in L_{\alpha v}((0,T),\ \frac{a}{\alpha},\ A_{p,q}^s(\mathbb{R}^n)). \tag{3.2.18}$$

Then there is a constant $c>0$, independent of u_0 and f, such that

$$\|W(\cdot,t)|A_{p,q}^{s+d}(\mathbb{R}^n)\| \ \le\ c\,t^{-\frac{d-\alpha g}{2\alpha}}\|u_0|A_{p,q}^{s+\alpha g}(\mathbb{R}^n)\| \tag{3.2.19}$$

$$+\ c\,t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t \tau^{av}\|f(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v} \tag{3.2.20}$$

for all t with $0<t<T$ (with the usual modification if $v=\infty$).

Proof. We apply Theorem 3.2.2 with $s+\alpha g$ in place of s and $d-\alpha g$ in place of d. Thus,

$$\|W_t^\alpha u_0|A_{p,q}^{s+d}(\mathbb{R}^n)\| \le c\,t^{-\frac{d-\alpha g}{2\alpha}}\|u_0|A_{p,q}^{s+\alpha g}(\mathbb{R}^n)\|. \tag{3.2.21}$$

For the second summand in (3.2.16) we obtain

$$\begin{aligned}
\|\int_0^t W_{t-\tau}^\alpha f(\cdot,\tau)\mathrm{d}\tau|A_{p,q}^{s+d}(\mathbb{R}^n)\| &\le \int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\tau^{-\frac{a}{\alpha}}\tau^{\frac{a}{\alpha}}\|f(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|\mathrm{d}\tau \\
\le\ c\left(\int_0^t (t-\tau)^{-\frac{d(\alpha v)'}{2\alpha}}\tau^{-\frac{a}{\alpha}(\alpha v)'}\mathrm{d}\tau\right)^{1/(\alpha v)'} &\left(\int_0^t \tau^{av}\|f(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v} \\
\le\ c\,t^{1-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}&\left(\int_0^t \tau^{av}\|f(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}
\end{aligned}$$

using Theorem 3.2.2, Hölder's inequality with exponent αv as well as the assumptions $a < \alpha - 1/v$ and $d < 2(\alpha - 1/v)$. □

Note, that it was immaterial to replace 1 in Theorem 3.2.2 by any fixed $T>0$.

3.3 Existence and uniqueness

In this subsection we prove our main result, that is the existence and uniqueness of mild and strong solutions of

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{3.3.1}$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $Du^2 = \sum_{j=1}^{n} \partial_j u^2$.

Interpretation of the nonlinear term Du^2 as inhomogeneity allows us to apply Proposition 3.2.3 to the operator T_{u_0} defined in (3.0.2). Then we combine the previous estimates with a fixed point argument for T_{u_0}. We call solutions $u \in L_1^{\mathrm{loc}}((0,T) \times \mathbb{R}^n)$ of (3.3.1), which are a fixed point of T_{u_0}, mild. In addition to uniqueness (local in time, $0 < t < T$) we look for strong solutions. A solution u is called strong if it is mild and belongs to the space $C([0,T), A^{s_0}_{p,q}(\mathbb{R}^n))$ for all intial data $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$. For more detailed explanations in this context we refer to [5] and [23]. This is the understanding of Theorem 3.3.5 below. The notation itself goes back to [4], [19] and has been used in the context of Navier-Stokes equations, that is in the vector-valued case for $\alpha = 1$ in [17] and [20].

3.3.1 Some preliminary considerations

We have to clarify first to which extend we can apply Proposition 2.3.4 when we set $\omega := u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$ and $f := Du^2$, $u \in L_v((0,T), b, A^s_{p,q}(\mathbb{R}^n))$. The choice of parameters will be specified later on. We consider $W(x,t)$ as defined in (3.2.16) and formulate the necessary conditions so that the first summand belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$.

Lemma 3.3.1. Let $1 \leq p, q \leq \infty$ ($p < \infty$ for F- spaces), $\alpha \in \mathbb{N}$, $n \geq 2$, $-2\alpha < s_0$ and $T > 0$. Let

$$W^\alpha(x,t) = \begin{cases} W_t^\alpha u_0(x), & x \in \mathbb{R}^n,\ t \in (0,T), \\ 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R} \setminus (0,T). \end{cases} \tag{3.3.2}$$

Then $W^\alpha(x,t)$ belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ for all $u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n)$.

Proof. We choose some $d \geq 0$ which satisfies $-s_0 < d < 2\alpha$. Then $A^{s_0+d}_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$ by Theorem 1.4.1 and Corollary 1.4.2. Then we apply Theorem 3.2.2 and obtain for any compact subset $\mathcal{K} \subset \mathbb{R}^{n+1}$

$$\begin{aligned}
\int_{\mathcal{K}} |W^\alpha(x,t)| \, \mathrm{d}(x,t) &\le \int_0^R \int_{|x|\le R} |W_t^\alpha u_0(x)| \, \mathrm{d}x \, \mathrm{d}t \\
&\le c \int_0^R \|W_t^\alpha u_0 | L_p(\mathbb{R}^n)\| \, \mathrm{d}t \\
&\le c \int_0^R \|W_t^\alpha u_0 | A_{p,q}^{s_0+d}(\mathbb{R}^n)\| \, \mathrm{d}t \\
&\le c \int_0^R t^{-d/2\alpha} \|u_0 | A_{p,q}^{s_0}(\mathbb{R}^n)\| \, \mathrm{d}t \qquad (3.3.3) \\
&\le c \, R^{-d/2\alpha+1}
\end{aligned}$$

with an appropriately chosen $R > 0$ because of $d < 2\alpha$. The first estimate is obtained by means of Hölder's inequality using the fact that $W_t^\alpha u_0 \in C^\infty(\mathbb{R}^n)$. □

Now we consider the second summand of (3.2.16). We investigate under which conditions the convolution of G^α and u^2 exists and belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ (after extension to $\mathbb{R}^{n+1}$ by zero).

Lemma 3.3.2. Let $1 \le p, q \le \infty$ ($p < \infty$ for F- spaces) and $s \ge \frac{n}{p}$ such that $A_{p,q}^s(\mathbb{R}^n)$ is a multiplication algebra. Let $u \in L_{2v}((0,T),\, b,\, A_{p,q}^s(\mathbb{R}^n))$ with $T > 0$, $1 < v \le \infty$ and $-\infty < 2b < 1 - \frac{1}{v}$.

(i) Let

$$U(x,t) = \begin{cases} u(x,t), & x \in \mathbb{R}^n,\ t \in (0,T), \\ 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R} \setminus (0,T). \end{cases} \qquad (3.3.4)$$

Then $U^2(x,t) \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$.

(ii) Let

$$H(x,t) = \begin{cases} \int\limits_0^t \int_{\mathbb{R}^n} G^\alpha(x-y, t-\tau) u^2(y,\tau) \, \mathrm{d}y \, \mathrm{d}\tau, & x \in \mathbb{R}^n,\ 0 < t < T \\ \int\limits_0^T \int_{\mathbb{R}^n} G^\alpha(x-y, t-\tau) u^2(y,\tau) \, \mathrm{d}y \, \mathrm{d}\tau, & x \in \mathbb{R}^n,\ t \ge T \\ 0, & x \in \mathbb{R}^n,\ t \le 0. \end{cases}$$

Then $H \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $H = G^\alpha * U^2$ in $D'(\mathbb{R}^{n+1})$.

Proof. **Step 1.** We show (i). It holds

$$\|u^2|L_v((0,T),\,2b,\,A^s_{p,q}(\mathbb{R}^n))\| = \left(\int_0^T t^{2bv}\|u^2(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|^v \mathrm{d}t\right)^{1/v}$$

$$\leq c\left(\int_0^T t^{2bv}\|u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|^{2v}\mathrm{d}t\right)^{1/v} = c\|u|L_{2v}((0,T),\,b,\,A^s_{p,q}(\mathbb{R}^n))\|^2.$$

Applying Lemma 2.3.5 with $2b$ in place of b this yields the assertion.
Step 2. Let $0 < t \leq T$. Then we obtain for any compact subset $\mathcal{K} \subset \mathbb{R}^{n+1}$

$$\int_{\mathcal{K}} |H(x,t)|\,\mathrm{d}(x,t) \leq \int_0^R \int_{|x|\leq R} |H(x,t)|\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^R \int_{|x|\leq R}\int_0^t |W^\alpha_{t-\tau}u^2(x,\tau)|\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t.$$

Application of Theorem 1.4.3 and Theorem 3.2.2 with some $0 \leq d < 2\alpha\left(1-\frac{1}{v}\right)$ yields

$$\begin{aligned}\int_{\mathcal{K}} |H(x,t)|\mathrm{d}(x,t) &\lesssim \int_0^R\int_0^t \|W^\alpha_{t-\tau}u^2(\cdot,\tau)|A^{s+d}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t\\ &\lesssim \int_0^R\int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\|u^2(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t\\ &\lesssim \int_0^R\int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\tau^{-2b}\tau^{2b}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\mathrm{d}\tau\,\mathrm{d}t.\end{aligned}$$

Now it follows similar to the proof of Proposition 3.2.3

$$\begin{aligned}\int_{\mathcal{K}} |H(x,t)|\mathrm{d}(x,t) &\lesssim \left(\int_0^R t^{1-\frac{1}{v}-2b-\frac{d}{2\alpha}}\mathrm{d}t\right)\|u|L_{2v}((0,T),\,b,\,A^s_{p,q}(\mathbb{R}^n))\|^2\\ &\sim R^{2-\frac{1}{v}-2b-\frac{d}{2\alpha}}\|u|L_{2v}((0,T),\,b,\,A^s_{p,q}(\mathbb{R}^n))\|^2.\end{aligned}$$

A similar calculation leads for $t \geq T$ to

$$\begin{aligned}\int_{\mathcal{K}} |H(x,t)|\mathrm{d}(x,t) &= \int_0^R\int_{|x|<R}\int_0^T |W^\alpha_{t-\tau}u^2(x,\tau)|\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \int_0^R\int_0^T (t-\tau)^{-\frac{d}{2\alpha}}\tau^{-2b}\tau^{2b}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\mathrm{d}\tau\,\mathrm{d}t\\ &\lesssim \int_0^R\int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\tau^{-2b}\tau^{2b}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\mathrm{d}\tau\,\mathrm{d}t\\ &\lesssim R^{2-\frac{1}{v}-2b-\frac{d}{2\alpha}}\|u|L_{2v}((0,T),\,b,\,A^s_{p,q}(\mathbb{R}^n))\|^2.\end{aligned}$$

Hence, $H \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$. $\square$

Lemma 3.3.3. Let $n \geq 2$, $\alpha \in \mathbb{N}$, $1 \leq p, q \leq \infty$ ($p < \infty$ for F- spaces) and s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra. Let

$$0 < \lambda < g \leq 1, \;\; \frac{2}{\alpha} < v \leq \infty, \;\; a = \alpha - \frac{1}{v} - \alpha\lambda, \tag{3.3.5}$$

and let $u_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$. Further assume that $u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))$ is a fixed point of T_{u_0} as defined in (3.0.2). Then it holds

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2$$

in the sense of $D'(\mathbb{R}^n \times (0,T))$.

Proof. Since $u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))$ is a fixed point of (3.0.2) it holds

$$u(x,t) = W^\alpha_t u_0(x) + \int_0^t W^\alpha_{t-\tau} Du^2(\cdot,\tau) \mathrm{d}\tau. \tag{3.3.6}$$

Here

$$W^\alpha_{t-\tau} Du^2(\cdot,\tau) = (G^\alpha_{t-\tau} * Du^2)(\cdot,\tau), \quad 0 < \tau < t$$

stands for the convolution of $G^\alpha_{t-\tau} \in S(\mathbb{R}^n)$ and $Du^2 \in S'(\mathbb{R}^n)$. Let U be the extension of u to $\mathbb{R}^n \times (\mathbb{R} \setminus (0,T))$ by zero as defined in (3.3.4). From Lemma 3.3.2 we know that the convolution of G^α and U^2 exists in the sense of $D'(\mathbb{R}^{n+1})$ and belongs to $L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})$. By Theorem 2.1.8 then also $D(G^\alpha * U^2) = (DG^\alpha) * U^2 = G^\alpha * DU^2$ exists. Let

$$\begin{aligned} k(x,t) &:= \int_0^t W^\alpha_{t-\tau} Du^2(x,\tau)\,\mathrm{d}\tau = \int_0^t (DG^\alpha_{t-\tau}) * u^2(x,\tau)\,\mathrm{d}\tau \\ &= \int_0^t \int_{\mathbb{R}^n} (DG^\alpha)(x-y, t-\tau) u^2(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \end{aligned}$$

for $0 < t < T$ and

$$\begin{aligned} K(x,t) &:= \int_{-\infty}^\infty \int_{\mathbb{R}^n} (DG^\alpha)(x-y,t-\tau) U^2(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &= \begin{cases} k(x,t), & x \in \mathbb{R}^n,\, 0 < t < T \\ \int\limits_0^T \int_{\mathbb{R}^n} (DG^\alpha)(x-y,t-\tau) u^2(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau, & x \in \mathbb{R}^n,\, t \geq T \\ 0, & x \in \mathbb{R}^n,\, t \leq 0. \end{cases} \end{aligned}$$

Combining (3.3.6), Lemma 2.3.5 and Lemma 3.3.1 we find that $K \in L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})$. Moreover $U^2 \in L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})$ by Lemma 3.3.2 and $DG^\alpha \in L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})$ as in Step 1 of Proposition 2.2.1.

Hence, by Proposition 2.3.2 it holds that

$$(\partial_t + (-\Delta_x)^\alpha)K = D(\partial_t + (-\Delta_x)^\alpha)(G^\alpha * U^2) = DU^2 \quad \text{in} \quad D'(\mathbb{R}^{n+1}). \tag{3.3.7}$$

By (3.3.6) we have

$$(\partial_t + (-\Delta_x)^\alpha)(u - W_t^\alpha u_0) = Du^2 \quad \text{in} \quad D'(\mathbb{R}^n \times (0,T)). \tag{3.3.8}$$

Thus by (2.3.10),

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2 \quad \text{in} \quad D'(\mathbb{R}^n \times (0,T)). \tag{3.3.9}$$

□

Remark 3.3.4. In particular Lemma 3.3.3 justifies the reformulation of (3.3.1) into the fixed point problem (3.0.2).

3.3.2 Main result

Theorem 3.3.5. *Let $n \geq 2$, $\alpha \in \mathbb{N}$, $1 \leq p,q \leq \infty$ ($p < \infty$ for F- spaces) and s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra.*

(i) *Let*

$$0 < \lambda < g \leq 1, \ \frac{2}{\alpha} < v \leq \infty, \ a = \alpha - \frac{1}{v} - \alpha\lambda \tag{3.3.10}$$

and let $u_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$ for the initial data. Then there exists a number $T > 0$ such that

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \mathbb{R}^n \end{aligned} \tag{3.3.11}$$

has a unique mild solution

$$u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T)).$$

(ii) *If, in addition, $p < \infty$, $q < \infty$ and*

$$\frac{g}{2} \leq \lambda < g \leq 1 \ \text{ if } \ v < \infty \quad \text{ and } \quad \frac{g}{2} < \lambda < g \leq 1 \ \text{ if } \ v = \infty \tag{3.3.12}$$

then the above solution is strong, that means $u \in C([0,T), A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n))$.

Proof. **Step 1.** We assume $v < \infty$. The main idea is to apply the estimates of Proposition 3.2.3 with $s - \alpha$ in place of s, $d = \alpha$ and

$$f = Du^2 \in L_{\alpha v}((0,T),\, \frac{a}{\alpha},\, A^{s-1}_{p,q}(\mathbb{R}^n)). \tag{3.3.13}$$

We ask for a fixed point of (3.0.2), i.e.

$$T_{u_0}u(x,t) = W^\alpha_t u_0(x) + \int_0^t W^\alpha_{t-\tau} Du^2(x,\tau)\mathrm{d}\tau \tag{3.3.14}$$

in $L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n))$. Then we obtain for $0 < t \le T$

$$\begin{aligned} \|T_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| &\le c\, t^{-\frac{\alpha-\alpha g}{2\alpha}} \|u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\ &\quad + c\, t^{1-\frac{1}{\alpha v}-\frac{\alpha}{2\alpha}-\frac{a}{\alpha}} \left(\int_0^t \tau^{av} \|Du^2(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|^{\alpha v} \mathrm{d}\tau \right)^{1/\alpha v}. \end{aligned} \tag{3.3.15}$$

We enlarge the integral extending the limits from $(0,t)$ to $(0,T)$ and multiply both sides with $t^{\frac{a}{2\alpha}}$. Raising to the power of $2\alpha v$ and integrating over $(0,T)$ yields

$$\begin{aligned} \int_0^T t^{av} \|T_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}t \;&\le\; c \int_0^T t^{(-\alpha+\alpha g+a)v}\, \mathrm{d}t\, \|u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|^{2\alpha v} \\ + c \int_0^T t^{\alpha v - 2 - av}\, \mathrm{d}t &\left(\int_0^T \tau^{av} \|Du^2(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau \right)^2 \\ \le c\,T^\delta \|u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|^{2\alpha v} &+ c\,T^\varkappa \left(\int_0^T \tau^{av} \|Du^2(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau \right)^2 \end{aligned} \tag{3.3.16}$$

where

$$\delta = (-\alpha + \alpha g + a)v + 1 = \alpha(g - \lambda)v > 0 \tag{3.3.17}$$

since $g > \lambda$ and

$$\varkappa = \alpha v - 1 - av = \alpha\lambda v > 0 \tag{3.3.18}$$

since $\lambda > 0$. Now we use the embedding $A^{s-1}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{s-\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \ge 1$, that D is a bounded mapping from $A^s_{p,q}(\mathbb{R}^n)$ to $A^{s-1}_{p,q}(\mathbb{R}^n)$ and the fact that $A^s_{p,q}(\mathbb{R}^n)$ is assumed to be a multiplication algebra. It follows that

$$\begin{aligned} &\int_0^T t^{av} \|T_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}t \\ &\le c\,T^\delta \|u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|^{2\alpha v} + c\,T^\varkappa \left(\int_0^T \tau^{av} \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau \right)^2. \end{aligned} \tag{3.3.19}$$

Thus, we see that if $T > 0$ for given u_0 is chosen sufficiently small then T_{u_0} maps the unit ball U_T in $L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n))$ into itself. To show that $T_{u_0} : U_T \mapsto U_T$ is a contraction, consider $u, v \in U_T$. A similar calculation as above gives

$$\begin{aligned}
&\|T_{u_0}u(\cdot,t) - T_{u_0}v(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\le c\, t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}} \left(\int_0^t \tau^{av}\|u^2(\cdot,\tau) - v^2(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v} \qquad (3.3.20)\\
&\le c\, t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}} \left(\int_0^t \tau^{av}\|u(\cdot,\tau) - v(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{\alpha v}\|u(\cdot,\tau) + v(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}.
\end{aligned}$$

Let temporarily $X^s_T = L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n))$. Applying Hölder's inequality with exponent 2 and using $u, v \in U_T$ this yields

$$\|T_{u_0}u - T_{u_0}v|X^s_T\| \le c\, T^{\frac{\varkappa}{2\alpha v}}\|u - v|X^s_T\|\|u + v|X^s_T\| \qquad (3.3.21)$$

with the same $\varkappa$ as is (3.3.18). The second norm can be estimated by means of Minkowski's inequality. Thus, $T_{u_0} : U_T \mapsto U_T$ is a contraction if $T > 0$ is small enough. Since we deal with Banach spaces we have shown that T_{u_0} has a unique fixed point in U_T which is according to Lemma 3.3.3 a solution of (3.3.1).

Step 2. So far we know that (3.3.1) has a unique mild solution in U_T. To extend this assertion to the whole space, we require a certain amount of preparation. In particular, we show first that the solution u is a C^∞-function with respect to space and time. Let again be $X^s_T = L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n))$. We start with the smoothness with respect to the space variable. To this end, we apply Proposition 3.2.3 with $s - \alpha$ in place of s and $d = \alpha + \eta$ with some $\eta > 0$ such that $\alpha + \eta < 2(\alpha - \frac{1}{v})$, which is possible since $2 < \alpha v$, and obtain

$$\begin{aligned}
\|T_{u_0}u(\cdot,t)|A^{s+\eta}_{p,q}(\mathbb{R}^n)\| &\le c\, t^{-\frac{(\alpha+\eta)-\alpha g}{2\alpha}}\|u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&\quad + c\, t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{\eta}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t \tau^{av}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}. \qquad (3.3.22)
\end{aligned}$$

The idea is to iterate this argument whereas each iteration step generates a solution of (3.3.1) which is smoother than the previous one. Therefore we need smoother initial data. Thus, we apply (3.3.15) - (3.3.19) with initial data $u_\varepsilon(x) = u(x,\varepsilon)$ at some $\varepsilon > 0$. Since $T_{u_0}u = u$ for a solution u of (3.3.1) we have $u_\varepsilon(x) \in A^{s+\eta}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{s+\eta-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$.

Replacing $s-\alpha$ by $s-\alpha+\eta$ yields

$$\begin{aligned}\|T_{u_\varepsilon}u(\cdot,t)|A^{s+2\eta}_{p,q}(\mathbb{R}^n)\| &\leq c\,t^{-\frac{(\alpha+\eta)-\alpha g}{2\alpha}}\|u_\varepsilon|A^{s+\eta-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\ &\quad + c\,t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{\eta}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t \tau^{av}\|Du^2(\cdot,\tau)|A^{s+\eta-\alpha}_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v} \\ &\leq c\,t^{-\frac{(\alpha+\eta)-\alpha g}{2\alpha}}\|u_\varepsilon|A^{s+\eta-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\ &\quad + c\,t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{\eta}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t \tau^{av}\|u(\cdot,\tau)|A^{s+\eta}_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{1/\alpha v} \qquad (3.3.23)\end{aligned}$$

since $A^{s+\eta}_{p,q}(\mathbb{R}^n)$ is multiplication algebra, too. In particular, we see that the exponents remain the same. We proceed as in Step 1 and enlarge the integral extending the limits to $(0,T)$. Raising to the power of $2\alpha v$ and integrating over $(0,T)$ leads after the k-th iteration to

$$\|T_{u_\varepsilon}u|X^{s+k\eta}_T\| \leq c\,T^{\frac{\delta'}{2\alpha v}}\|u_\varepsilon|A^{s+(k-1)\eta-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| + c\,T^{\frac{\widetilde{\varkappa}}{2\alpha v}}\|u|X^{s+(k-1)\eta}_T\|$$

where

$$\delta' = \delta - \eta v > 0 \quad \text{and} \quad \widetilde{\varkappa} = \varkappa - \eta v > 0$$

with δ, $\varkappa$ as in (3.3.17), (3.3.18) and a sufficiently small $\eta > 0$. In doing so we obtain after the k-th step a solution u of (3.3.1) which belongs to some function space $A^{s+k\eta}_{p,q}(\mathbb{R}^n)$. Now we conclude with the embedding $A^{s+k\eta}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s+k\eta-\frac{n}{p}}_{\infty,\infty} = \mathcal{C}^{s+k\eta-\frac{n}{p}}(\mathbb{R}^n)$ for any $k\in\mathbb{N}$ that u is a C^∞-function with respect to the space variable.

Step 3. We show that the solution u is a C^∞-function with respect to t in any interval (t_0,t_1) with $0<t_0<t_1<T$. From Step 1 we know that $u\in L_\infty((0,T),\frac{a}{2\alpha},A^s_{p,q}(\mathbb{R}^n))$, i.e.

$$\sup_{t\in(t_0,t_1)} t^{\frac{a}{2\alpha}}\|u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| < \infty.$$

From this it follows

$$\begin{aligned}\sup_{t\in(t_0,t_1)}\|u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| &= \sup_{t\in(t_0,t_1)} t^{-\frac{a}{2\alpha}}t^{\frac{a}{2\alpha}}\|u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\ &\leq t_0^{-\frac{a}{2\alpha}}\sup_{t\in(t_0,t_1)} t^{\frac{a}{2\alpha}}\|u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| < \infty\end{aligned}$$

independent of s. In combination with Step 2 this leads to

$$\sup_{x\in\Omega,\,t\in(t_0,t_1)}|\mathrm{D}^\beta_x u(x,t)| \leq c_\beta \quad \text{for all } \beta\in\mathbb{N}^n_0.$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Since u satisfies

$$(\partial_t + (-\Delta_x^\alpha)u = Du^2, \tag{3.3.24}$$

as we have shown in Lemma 3.3.3, this is also true for the distributional derivative $\partial_t u$. Thus we obtain iteratively

$$\sup_{x\in\Omega,\, t\in(t_0,t_1)} |\frac{\partial^k}{\partial t^k} \mathrm{D}_x^\beta u(x,t)| \le c_\beta, \quad \text{for all } \beta \in \mathbb{N}_0^n,$$

hence, $\frac{\partial^k}{\partial t^k}\mathrm{D}_x^\beta u \in L_\infty(\Omega \times (t_0,t_1))$. Application of Sobolevs embedding theorem leads to the desired result for any $k \in \mathbb{N}$, $\beta \in \mathbb{N}_0$.

Step 4. To extend the uniqueness to the whole space let $u \in U_T$ be the above solution and $v \in X_T^s$ a second solution. From (3.3.20) - (3.3.21) we know that

$$\|T_{u_0}u(\cdot,t) - T_{u_0}v(\cdot,t)|A_{p,q}^s(\mathbb{R}^n)\| \qquad \le c\, t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{a}{\alpha}} \|u-v|X_{T_0}^S\|\|u+v|X_{T_0}^S\| \tag{3.3.25}$$

for any $0 < t \le T_0 \le T$. We apply Minkowski's inequality and use $u \in U_T$. Then we obtain for $0 < T_0 \le T$ and the same $\varkappa > 0$ as in (3.3.18)

$$\|u-v|X_{T_0}^s\| \le c\, T_0^\varkappa\,(1 + \|v|X_T^s\|)\|u-v|X_{T_0}^s\|. \tag{3.3.26}$$

If we choose $T_0 > 0$ small enough such that $c\, T_0^\varkappa\,(1 + \|v|X_T^s\|) < 1$ it follows that $u(\cdot,t) = v(\cdot,t)$ for any $t \in (0,T_0]$. Denote

$$T_1 := \sup\{t \in (0,T\,] : u(\cdot,t) = v(\cdot,t)\}$$

and assume $T_1 < T$. Because of the continuity of u and v, T_1 is the maximum with this property. Now we take $u(\cdot,T_1) \in A_{p,q}^s(\mathbb{R}^n) \hookrightarrow A_{p,q}^{s-\alpha+\alpha g}$ as new initial value and proceed as in the previous steps until (3.3.26) inclusively. There exists a unique solution $\widetilde{u}$ in a neighbourhood $U_\delta(T_1)$ with $\widetilde{u}(\cdot,T_1) = u(\cdot,T_1)$. Since it holds that $\widetilde{u}(\cdot,t) = u(\cdot,t)$ for all $t \in (0,T_1] \cap U_\delta(T_1)$ we have extended u to some interval $(0,T_2]$ with $T_1 < T_2$. Thus, we have prolongated $u(\cdot,t) - v(\cdot,t) = 0$ to some interval $(0,T_2]$ where $T_1 < T_2 \le T$ which contradicts the assumption. This proves the uniqueness in $L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A_{p,q}^s(\mathbb{R}^n))$.

Step 5. We show first, that the continuity up to $t = 0$ depends only on $W_t^\alpha u_0$ and u_0.

$$\begin{aligned}&\|u(\cdot,t) - u_0|A_{p,q}^{s-\alpha+\alpha g}(\mathbb{R}^n)\| \\ &\le c\,\|W_t^\alpha u_0 - u_0|A_{p,q}^{s-\alpha+\alpha g}(\mathbb{R}^n)\| + \int_0^t \|W_{t-\tau}^\alpha Du^2(\cdot,\tau)|A_{p,q}^{s-\alpha+\alpha g}(\mathbb{R}^n)\|\mathrm{d}\tau\end{aligned} \tag{3.3.27}$$

To the second summand, which we denote by (I), we apply Theorem 3.2.2 with $d = \alpha g$, $s-\alpha$ in place of s and fixed $t \in (0,T)$ as follows

$$
\begin{aligned}
\text{(I)} &\le c \int_0^t (t-\tau)^{-\frac{\alpha g}{2\alpha}} \|Du^2(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \mathrm{d}\tau \\
&\le c \int_0^t (t-\tau)^{-\frac{g}{2}} \|u^2(\cdot,\tau)|A^{s}_{p,q}(\mathbb{R}^n)\| \mathrm{d}\tau \\
&\le c \int_0^t (t-\tau)^{-\frac{g}{2}} \tau^{-\frac{a}{\alpha}} \tau^{\frac{a}{\alpha}} \|u(\cdot,\tau)|A^{s}_{p,q}(\mathbb{R}^n)\|^2 \mathrm{d}\tau \\
&\le c \left(\int_0^t (t-\tau)^{-\frac{g}{2}(av)'} \tau^{-\frac{a(\alpha v)'}{\alpha}} \mathrm{d}\tau \right)^{\frac{1}{(\alpha v)'}} \left(\int_0^t \tau^{av} \|u(\cdot,\tau)|A^{s}_{p,q}(\mathbb{R}^n)\|^{2\alpha v} \mathrm{d}\tau \right)^{\frac{1}{\alpha v}} \\
&= t^{-\frac{g}{2}-\frac{a}{\alpha}+1-\frac{1}{\alpha v}} \left(\int_0^t \tau^{av} \|u(\cdot,\tau)|A^{s}_{p,q}(\mathbb{R}^n)\|^{2\alpha v} \mathrm{d}\tau \right)^{\frac{1}{\alpha v}}. \qquad (3.3.28)
\end{aligned}
$$

Assuming $v < \infty$ and letting t tend to zero then (3.3.28) tends to zero if $-\frac{g}{2} - \frac{a}{\alpha} + 1 - \frac{1}{\alpha v} \ge 0$. That means, one can consider solutions $u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))$ with parameters $a \le \alpha - \frac{\alpha g}{2} - \frac{1}{v}$. If $v = \infty$ one has to choose $a < \alpha - \frac{\alpha g}{2} - \frac{1}{v}$ since the integral is substituted by the supremum. This is satisfied if $\lambda \ge \frac{g}{2}$ and $\lambda > \frac{g}{2}$, respectively.
Concerning the first summand in (3.3.27) we obtain

$$
\begin{aligned}
W^\alpha_t u_0(x) - u_0(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} t^{-n/2\alpha} \left(e^{-|\xi|^{2\alpha}}\right)^\vee \left(\frac{x-y}{t^{1/2\alpha}}\right) u_0(y)\mathrm{d}y - u_0(x) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z)[u_0(x - t^{1/2\alpha}z) - u_0(x)]\mathrm{d}z. \qquad (3.3.29)
\end{aligned}
$$

Thus, since $p,q \ge 1$

$$
\begin{aligned}
&\|W^\alpha_t u_0 - u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \int_{|z|\le N} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z)\|u_0(x - t^{1/2\alpha}z) - u_0(x)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|\mathrm{d}z \\
&+ \int_{|z|> N} \left(e^{-|\xi|^{2\alpha}}\right)^\vee (z)\|u_0(x - t^{1/2\alpha}z) - u_0(x)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|\mathrm{d}z. \qquad (3.3.30)
\end{aligned}
$$

Since $u_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$ the second integral is smaller than ε, independent of x and t, if we choose a sufficiently large $N > 0$. Fixing this N, also the first integral is smaller than ε, provided that t is appropriately small, $0 < t \le t_0(\varepsilon)$, and $\max(p,q) < \infty$. Thus,

$$
\|W^\alpha_t u_0 - u_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \le 2\varepsilon \quad \text{if } t \le t_0(\varepsilon). \qquad (3.3.31)
$$

Since ε can be choosen arbitrarily small we have shown the continuity of $u(\cdot,t)$ up to $t = 0$ in the Banach space $A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$. Hence the solution u is strong in the prescribed sense. □

Remark 3.3.6. The classical case $\alpha = 1$ of Theorem 3.3.5 is essentially covered by [38, Theorems 4.10, 4.14, pp. 119, 121]

3.4 Stability and well-posedness

In addition to the results of the previous part one may ask for stability of a solution. That means small perturbations of the initial data cause small deviations of the solution. A solution is called locally stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ and a time $T > 0$ such that for all $0 < t < T$ holds

$$\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s_0}_{p,q}(\mathbb{R}^n)\| \leq \varepsilon \tag{3.4.1}$$

if

$$\|u^1_0 - u^2_0|A^{s_0}_{p,q}(\mathbb{R}^n)\| \leq \delta \tag{3.4.2}$$

whereas u_i is a solution (3.3.1) with initial data u^i_0, $i = 1, 2$. The problem (3.3.1) is called well-posed if there is a unique mild solution which is additionally strong and stable in the above sense.
It is sufficient to show the stability for $u \in L_\infty((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n))$. Recall that by construction of the solution as a fixed point of T_{u_0} we have $\|u|L_\infty((0,T),\ \frac{a}{2\alpha},\ A^s_{p,q}(\mathbb{R}^n))\| \leq 1$.

Theorem 3.4.1. *Let $n \geq 2$, $\alpha \in \mathbb{N}$, $1 \leq p,q \leq \infty$ ($p < \infty$ for F- spaces), s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra and a, g, λ as in Theorem 3.3.5 with $v = \infty$ whereas λ as in* (3.3.12). *Further, let u_i, $i = 1, 2$ be solutions of* (3.3.1) *obtained in Theorem 3.3.5 with initial data $u^i_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)$ in the corresponding time interval $(0, T_i)$. Then*

$$\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \leq c_0\|u^1_0 - u^2_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| + c_1\, t^{-\frac{g}{2}-\frac{a}{\alpha}+1} \tag{3.4.3}$$

for all $0 < t < T := \min(T_1, T_2)$. The constants $c_0 > 0$ and $c_1 > 0$ are independent of the initial data and t.

Proof. Let u_1, u_2 be two solutions of (3.3.1) with corresponding initial data u^1_0, u^2_0. Then it holds

$$\begin{aligned} &\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\ &\leq \|W^\alpha_t(u^1_0 - u^2_0)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| + \int_0^t \|W^\alpha_{t-\tau}(Du^2_1 - Du^2_2)(\cdot,\tau)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau. \end{aligned} \tag{3.4.4}$$

We apply Theorem 3.2.2 with $d = 0$ to the first summand and with $d = \alpha g$ and $s - \alpha$ in place of s to the second summand. This yields

$$\begin{aligned} &\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\ &\leq c_0\,\|u^1_0 - u^2_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| + c\int_0^t (t-\tau)^{-g/2}\|D(u^2_1 - u^2_2)(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau. \end{aligned} \tag{3.4.5}$$

By means of Minkowski's inequality we can split the norm in the second summand and estimate the terms with u_1 and u_2 separately. Let u be either u_1 or u_2. Then we obtain, using similar arguments as in the proof of Theorem 3.3.5,

$$\int_0^t (t-\tau)^{-g/2} \|Du^2(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \,\mathrm{d}\tau \lesssim \int_0^t (t-\tau)^{-g/2} \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2 \,\mathrm{d}\tau$$
$$= \int_0^t (t-\tau)^{-g/2} \tau^{-a/\alpha} (\tau^{a/2\alpha} \, \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|)^2 \,\mathrm{d}\tau$$
$$\leq \int_0^t (t-\tau)^{-g/2} \tau^{-a/\alpha} \,\mathrm{d}\tau \leq c_1 t^{-g/2-a/\alpha+1}, \quad 0 < t < T$$

because of $\|u|L_\infty((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))\| \leq 1$. The exponent is strictly positive due to the choice of the parameters. This proves (3.4.3). □

Now one can choose for any $\varepsilon > 0$ a suitable $\delta > 0$ and an appropriate $T > 0$ such that (3.4.1) follows from (3.4.2) for any $0 < t < T$. Hence, the solution of (3.3.1) obtained in Theorem 3.3.5 is stable and thus the problem well-posed if $\max(p,q) < \infty$.

4 Beyond multiplication algebras

In this chapter we consider the generalized nonlinear heat equation

$$\partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) = 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \tag{4.0.1}$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \tag{4.0.2}$$

with $0 < T \le \infty$, $2 \le n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $Du^2 = \sum\limits_{j=1}^{n} \partial_j u^2$ where the solution u belongs now with respect to the space variable to some spaces $A^s_{p,q}(\mathbb{R}^n)$ with $\frac{n}{p} - 1 < s < \frac{n}{p}$ and $s > 0$. This ensures that these spaces consist entirely of regular distributions. They are not multiplication algebras but still possess sufficient multiplication properties. As for the initial data u_0 we assume

$$u_0 \in A^{s_0}_{p,q}(\mathbb{R}^n) \quad \text{with } s - \alpha\left(1 - \frac{n}{p} + s\right) < s_0 \le s \tag{4.0.3}$$

such that we are within the supercritical case. Furthermore, we study the limiting case $s = \frac{n}{p} > 0$ when $A^s_{p,q}(\mathbb{R}^n)$ is not a multiplication algebra and the interesting case that the solution u belongs to $F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, $1 < p < \infty$, also in the framework of supercritical spaces.
The chapter is organized as follows. We start with some preliminary considerations on which spaces $A^s_{p,q}(\mathbb{R}^n)$ should be called critical, sub- and supercritical in the context of (4.0.1), (4.0.2). Then we deal with mapping properties of the nonlinearity Du^2 under the weaker conditions on the smoothness parameter s. As main result we show that (4.0.1), (4.0.2) has a unique strong and stable solution u in each of the above mentioned cases which is a fixed point of T_{u_0} as defined in (3.0.2).

4.1 Critical and supercritical spaces

Assume that $u = u(x,t)$ is a solution of

$$\partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - \sum_{i=1}^{n} \partial_j u^2(x,t) = 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \tag{4.1.1}$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \tag{4.1.2}$$

with $0 < T \le \infty$, $2 \le n \in \mathbb{N}$ and $\alpha \in \mathbb{N}$.

Proposition 4.1.1. *Let $\lambda > 0$ and $u_0 = u_0(x)$, $x \in \mathbb{R}^n$. Let $u^\lambda = u^\lambda(x,t)$, $x \in \mathbb{R}^n$ $0 \le t < T$ be a solution of*

$$\partial_t u^\lambda(x,t) + (-\Delta_x)^\alpha u^\lambda(x,t) - \sum_{j=1}^n \partial_j (u^\lambda)^2(x,t) = 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \quad (4.1.3)$$

$$u^\lambda(x,0) = \lambda^{-2\alpha+1}\, u_0(\lambda^{-1}x),\ \ x \in \mathbb{R}^n. \quad (4.1.4)$$

Then

$$u_\lambda(x,t) = \lambda^{2\alpha-1} u^\lambda(\lambda x, \lambda^{2\alpha} t), \qquad x \in \mathbb{R}^n,\ 0 \le t < \lambda^{-2\alpha}\, T \quad (4.1.5)$$

is a solution of

$$\partial_t u_\lambda(x,t) + (-\Delta_x)^\alpha u_\lambda(x,t) - \sum_{i=1}^n \partial_j u_\lambda^2(x,t) = 0, \qquad x \in \mathbb{R}^n,\ 0 < t < \lambda^{-2\alpha}\, T, \quad (4.1.6)$$

$$u_\lambda(x,0) = u_0(x),\ \ x \in \mathbb{R}^n. \quad (4.1.7)$$

In particular we see, that if u is a soluion of (4.1.1), (4.1.2) in $\mathbb{R}^n \times (0,T)$ then u_λ solves the same problem in $\mathbb{R}^n \times (0, \lambda^{-2\alpha}T)$.
Now we assume that u^λ is a solution of (4.1.3), (4.1.4) in $\mathbb{R}^n \times (0,T)$ and the initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}^n$ have to fulfill the additional restriction

$$u^\lambda(x,0) \in \dot{A}^s_{p,q}(\mathbb{R}^n) \qquad \text{with} \qquad \|u^\lambda(x,0)|\dot{A}^s_{p,q}(\mathbb{R}^n)\| \le \delta \quad (4.1.8)$$

with some $\delta > 0$. $\dot{A}^s_{p,q}(\mathbb{R}^n)$ denote the homogeneous spaces as introduced in Section 1.2. Recall that then

$$\|f(\lambda\cdot)|\dot{A}^s_{p,q}(\mathbb{R}^n)\| = \lambda^{s-\frac{n}{p}} \|f|\dot{A}^s_{p,q}(\mathbb{R}^n)\|, \quad f \in \dot{A}^s_{p,q}(\mathbb{R}^n),\ \lambda > 0. \quad (4.1.9)$$

Now the question arises which condition we have to impose on u_0 such that u_λ according to (4.1.5) - (4.1.7) is again a solution of (4.1.1), (4.1.2) with the same initial data in $\mathbb{R}^n \times (0, \lambda^{-2\alpha}T)$. For this purpose one has to solve (4.1.3), (4.1.4) under the above assumptions, that is

$$\begin{aligned} \lambda^{-2\alpha+1-s+\frac{n}{p}} \|u_0|\dot{A}^s_{p,q}(\mathbb{R}^n)\| &= \lambda^{-2\alpha+1}\|u_0(\lambda^{-1}\cdot)|\dot{A}^s_{p,q}(\mathbb{R}^n)\| && (4.1.10)\\ &= \|u^\lambda(\cdot,0)|\dot{A}^s_{p,q}(\mathbb{R}^n)\| \le \delta. && (4.1.11) \end{aligned}$$

In other words, if

$$\|u_0|\dot{A}^s_{p,q}(\mathbb{R}^n)\| \le \delta\lambda^{2\alpha-1+s-\frac{n}{p}}, \quad \lambda > 0, \quad (4.1.12)$$

then u_λ solves (4.1.6), (4.1.7). This suggests to call spaces

$$A^s_{p,q}(\mathbb{R}^n),\ \dot{A}^s_{p,q}(\mathbb{R}^n), \quad 0 < p, q \le \infty,\ s = \frac{n}{p} - 2\alpha + 1 \quad (4.1.13)$$

critical. Assume that there exists for any initial data $u_0 \in \dot{A}^s_{p,q}(\mathbb{R}^n)$, $s = \frac{n}{p} - 2\alpha + 1$ with $\|u_0|\dot{A}^s_{p,q}(\mathbb{R}^n)\| \leq \delta$ for some $\delta > 0$ a solution of the Cauchy problem (4.1.1), (4.1.2) in $\mathbb{R}^n \times (0, T)$, $u(x, 0) = u_0(x)$. Then by the above considerations there exists a solution u_λ of (4.1.1), (4.1.2) in $\mathbb{R}^n \times (0, \lambda^{-2\alpha}T)$ for any $\lambda > 0$ with the same initial data, too. Assuming additionally uniqueness of this solution this yields to a global unique solution in $\mathbb{R}^n \times (0, \infty)$, since any $\lambda > 0$ is possible. In case of supercritical spaces, i.e for spaces

$$A^s_{p,q}(\mathbb{R}^n),\ \dot{A}^s_{p,q}(\mathbb{R}^n), \quad 0 < p, q \leq \infty,\ s > \frac{n}{p} - 2\alpha + 1, \tag{4.1.14}$$

one has the following situation. Assume that there exists for any initial data u_0 with (4.1.8), $\delta > 0$ and $s > \frac{n}{p} - 2\alpha + 1$ a solution of (4.1.1), (4.1.2) in $\mathbb{R}^n \times (0, T)$, $u(x, 0) = u_0(x)$. Then one has by (4.1.12) with $\lambda \to \infty$ for arbitrary large initial data u_0 a solution of (4.1.1), (4.1.2) in $\mathbb{R}^n \times (0, \lambda^{-2\alpha}T)$. That means the larger the initial data the smaller the time interval for which a local solution exists. Letting tend λ to zero one obtains a global solution under the condition that the initial data become arbitrarily small. For subcritical spaces $\dot{A}^s_{p,q}(\mathbb{R}^n)$ with $s < \frac{n}{p} - 2\alpha + 1$ by the above considerations one would obtain a global solution of (4.1.1), (4.1.2) for arbitrary large initial data. This is rather doubtful.

The most interesting case is the critical case since one is always interested in solutions which are global in time. In the case $\alpha = 1$, that is for the Navier-Stokes equations, spaces $A^s_{p,q}(\mathbb{R}^n)$ and $\dot{A}^s_{p,q}(\mathbb{R}^n)$ with $s = \frac{n}{p} - 1$ are called critical. This applies especially to the spaces

$$L_n(\mathbb{R}^n) \hookrightarrow \dot{B}^{\frac{n}{p}-1}_{p,\infty}(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}^{-1}(\mathbb{R}^n), \quad 0 < p < \infty. \tag{4.1.15}$$

In these cases one has the well-posedness assertion according to [21] for global solutions of the Navier-Stokes equations with a suitable small chosen $\delta > 0$.

Note that $\mathrm{BMO}^{-1}(\mathbb{R}^n) = \dot{F}^{-1}_{\infty,2}(\mathbb{R}^n)$ is the largest critical space where such existence results are available. For the remaining critical spaces

$$\dot{B}^{-1}_{\infty,q}(\mathbb{R}^n) \text{ and } \dot{F}^{-1}_{\infty,q}(\mathbb{R}^n) \quad \text{with } 2 < q \leq \infty \tag{4.1.16}$$

the situation is different. In particular [3] proved the ill-posedness of the Navier-Stokes problem with initial data in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$. For a more detailed overview we refer to [2], [3], [7], [5], [9], [13], [18], [23], [24], [45] amongst others and for more references to [37, pp. 191-194, 209-214]. In what follows we focus our investigations on supercritical inhomogeneous spaces $A^s_{p,q}(\mathbb{R}^n)$.

4.2 Main results

Before we can derive a result similar to the case when the underlying function spaces with respect to the space variable are multiplication algebras, i.e. for Theorem 3.3.5, we have to deal with mapping properties of the nonlinearity Du^2.

Proposition 4.2.1. *Let $0 < p \leq \infty$ ($p < \infty$ for F-spaces), $0 < q \leq \infty$ and let $0 < \frac{1}{p} - \frac{s}{n} = \frac{1}{r} < \frac{1}{2}$, $s > 0$.*

(i) *Then*

$$F^s_{p,q}(\mathbb{R}^n) \cdot F^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p_r,q}(\mathbb{R}^n), \quad \frac{1}{p_r} = \frac{1}{p} + \frac{1}{r}.$$

Furthermore

$$\|D(uv)|F^{s-1-n/r}_{p,q}(\mathbb{R}^n)\| \leq c\,\|u|F^s_{p,q}(\mathbb{R}^n)\|\|v|F^s_{p,q}(\mathbb{R}^n)\|, \quad u,v \in F^s_{p,q}(\mathbb{R}^n).$$

(ii) *If additionally $q \leq r$ then*

$$B^s_{p,q}(\mathbb{R}^n) \cdot B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p_r,q}(\mathbb{R}^n), \quad \frac{1}{p_r} = \frac{1}{p} + \frac{1}{r}.$$

Furthermore

$$\|D(uv)|B^{s-1-n/r}_{p,q}(\mathbb{R}^n)\| \leq c\,\|u|B^s_{p,q}(\mathbb{R}^n)\|\|v|B^s_{p,q}(\mathbb{R}^n)\|, \quad u,v \in B^s_{p,q}(\mathbb{R}^n).$$

Proof. To show (i) we apply Theorem 1.4.4 with p in place of p_1, p_r in place of p and Theorem 1.4.6 with the same p and p_r and additionally $s - \frac{n}{r}$ in place of s_1. Then we obtain

$$\begin{aligned}\|D(uv)|F^{s-1-n/r}_{p,q}(\mathbb{R}^n)\| &\leq c\,\|uv|F^{s-n/r}_{p,q}(\mathbb{R}^n)\| \leq c\,\|uv|F^s_{p_r,q}\|(\mathbb{R}^n) \\ &\leq c\,\|u|F^s_{p,q}(\mathbb{R}^n)\|\|v|F^s_{p,q}(\mathbb{R}^n)\|. \end{aligned} \tag{4.2.1}$$

The proof of part (ii) follows along the lines of part (i). □

Analogously to Section 3.3.1 we justify that under the conditions of Theorem 4.2.3 below Proposition 2.3.4 with $f = Du^2$ can be applied.

Lemma 4.2.2. Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $1 \leq p,q \leq \infty$ and $s > 0$. Let $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$ and $\infty > r > n$ ($p < \infty$ for F-spaces, $q \leq r$ for B-spaces). Let

$$\frac{2n}{r} < \lambda < g \leq 1 + \frac{n}{r}, \quad 0 \leq \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{r}\right), \quad a = \alpha\left(1 + \frac{n}{r}\right) - \frac{1}{v} - \alpha\lambda \tag{4.2.2}$$

and let $u_0 \in A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)$. Further assume that $u \in L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n))$ is a fixed point of T_{u_0} as defined in (3.0.2). Then it holds

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2$$

in the sense of $D'(\mathbb{R}^n \times (0,T))$.

Proof. Since u is a fixed point of T_{u_0} it can be written as

$$u(x,t) = W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T. \tag{4.2.3}$$

We consider the first summand. Let W^α be its extension by zero to $\mathbb{R}^n \times (\mathbb{R} \setminus (0,T))$ as defined in Lemma 3.3.1. To show that W^α belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ we check that $s_0 = s - \alpha\left(1 + \frac{n}{r}\right) + \alpha g > -2\alpha$. But this is satisfied since $r > n$.

As for the second summand let again U be the extension of u by zero to $\mathbb{R}^n \times (\mathbb{R} \setminus (0,T))$ as defined in (3.3.4). From Proposition 4.2.1 we know that $u^2(\cdot,\tau) \in A^s_{p_r,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n)$. From Lemma 2.3.5 and Lemma 3.3.2 it follows with αv in place of v and $\frac{a}{2\alpha}$ in place of b that $U^2 \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ if $u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))$. Furthermore, U belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$. To show that $H = G^\alpha * U^2$, where H has the meaning of Lemma 3.3.2, exists and belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$ we proceed analogously to Lemma 3.3.2. Hereby we restrict ourselves to the case $0 < t < T$. The result for $t \geq T$ is obtained similarly. We choose some $d \geq 0$ with $-s + \frac{n}{r} < d < 2\left(\alpha - \frac{1}{v}\right)$ which is possible since $s > 0$, $\frac{1}{v} < 1 \leq \alpha$ and

$$\frac{n}{r} < 1 - \frac{2}{\alpha v} \leq \alpha - \frac{2}{v} < 2\left(\alpha - \frac{1}{v}\right).$$

Using Theorem 3.2.2 and the fact that $W^\alpha_{t-\tau}u^2(\cdot,\tau) \in C^\infty(\mathbb{R}^n)$ we obtain for any compact subset $\mathcal{K} \subset \mathbb{R}^{n+1}$

$$\begin{aligned}
\int_{\mathcal{K}} |H(x,t)|\,\mathrm{d}x\,\mathrm{d}t &\lesssim \int_0^R \int_{|x|<R} \int_0^t |W^\alpha_{t-\tau}u^2(x,\tau)|\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \int_0^R \int_0^t \|W^\alpha_{t-\tau}u^2(\cdot,\tau)|L_p(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim \int_0^R \int_0^t \|W^\alpha_{t-\tau}u^2(\cdot,\tau)|A^{s-\frac{n}{r}+d}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim \int_0^R \int_0^t (t-\tau)^{-\frac{d}{2\alpha}} \|u^2(\cdot,\tau)|A^{s-\frac{n}{r}}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim \int_0^R \int_0^t (t-\tau)^{-\frac{d}{2\alpha}} \tau^{-\frac{a}{\alpha}} \tau^{\frac{a}{\alpha}} \|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\,\mathrm{d}\tau\,\mathrm{d}t \qquad (4.2.4)\\
&\lesssim R^{2-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}} \|u|L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))\|^2.
\end{aligned}$$

In (4.2.4) we used Proposition 4.2.1. Defining $K = D(G^\alpha * U^2) = G^\alpha * DU^2$ the remaining steps coincide exactly with those of Lemma 3.3.3. Hence,

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2 \quad \text{in} \quad D'(\mathbb{R}^n \times (0,T)).$$

$\square$

Theorem 4.2.3. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $s > 0$. Let $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$ and $\infty > r > n$ ($p < \infty$ for F-spaces, $q \leq r$ for B-spaces).*

(i) *Let*

$$\frac{2n}{r} < \lambda < g \leq 1 + \frac{n}{r}, \quad 0 \leq \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{r}\right), \quad a = \alpha\left(1 + \frac{n}{r}\right) - \frac{1}{v} - \alpha\lambda \quad (4.2.5)$$

and let $u_0 \in A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)$ for the initial data. Then there exists a number $T > 0$ such that

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= 0, \qquad \text{in } \mathbb{R}^n \times (0,T), && (4.2.6) \\ u(x,0) &= u_0(x), \quad \text{in } \mathbb{R}^n && (4.2.7) \end{aligned}$$

has a unique solution

$$u \in L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A_{p,q}^s(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T)).$$

(ii) *If, in addition, $p < \infty$, $q < \infty$ and*

$$\frac{g}{2} + \frac{n}{r} \leq \lambda < g \leq 1 + \frac{n}{r} \text{ if } v < \infty \text{ and } \frac{g}{2} + \frac{n}{r} < \lambda < g \leq 1 + \frac{n}{r} \text{ if } v = \infty \quad (4.2.8)$$

then the above solution is strong, that means $u \in C([0,T), A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n))$.

Proof. **Step 1.** We assume $v < \infty$. The idea of the proof is similar to that of Theorem 3.3.5. We apply Proposition 3.2.3 with $d = \alpha\left(1 + \frac{n}{r}\right)$, $s - d$ in place of s and

$$f = Du^2 \in L_{\alpha v}((0,T),\, \frac{a}{\alpha},\, A_{p,q}^{s-1-\frac{n}{r}}(\mathbb{R}^n)). \quad (4.2.9)$$

Then we obtain for $0 < t < T$

$$\begin{aligned} &\|T_{u_0}u(\cdot,t)|A_{p,q}^s(\mathbb{R}^n)\| \leq c\, t^{-\frac{\alpha(1+\frac{n}{r})-\alpha g}{2\alpha}} \|u_0|A_{p,q}^{s-\alpha(1+\frac{n}{r})+\alpha g}(\mathbb{R}^n)\| \\ &+ t^{1-\frac{1}{\alpha v}-\frac{\alpha(1+\frac{n}{r})}{2\alpha}-\frac{a}{\alpha}} \left(\int_0^t \tau^{av}\|Du^2(\cdot,\tau)|A_{p,q}^{s-\alpha(1+\frac{n}{r})}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}. \end{aligned} \quad (4.2.10)$$

We enlarge the integral extending the limits from $(0,t)$ to $(0,T)$ and multiply both sides with $t^{\frac{a}{2\alpha}}$. Raising to the power of $2\alpha v$ and integrating over $(0,T)$ yields

$$\begin{aligned} &\int_0^T t^{av}\|T_{u_0}u(\cdot,t)|A_{p,q}^s(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}t \\ &\leq c \int_0^T t^{(-\alpha\left(1+\frac{n}{r}\right)+\alpha g+a)v}\,\mathrm{d}t\,\|u_0|A_{p,q}^{s-\alpha(1+\frac{n}{r})+\alpha g}(\mathbb{R}^n)\|^{2\alpha v} \\ &+ c\int_0^T t^{(\alpha-\frac{2}{v}-\frac{n\alpha}{r}-a)v}\,\mathrm{d}t \left(\int_0^T \tau^{av}\|Du^2(\cdot,\tau)|A_{p,q}^{s-\alpha(1+\frac{n}{r})}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^2. \end{aligned} \quad (4.2.11)$$

Now we use the embedding $A_{p,q}^{s-\left(1+\frac{n}{r}\right)}(\mathbb{R}^n) \hookrightarrow A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)}(\mathbb{R}^n)$ and Proposition 4.2.1. Hence,

$$\begin{aligned} &\|T_{u_0}u|L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A_{p,q}^s(\mathbb{R}^n))\| \\ &\le c\,T^{\frac{\delta}{2\alpha v}}\|u_0|A_{p,q}^{s-\alpha(1+\frac{n}{r})+\alpha g}(\mathbb{R}^n)\| + c\,T^{\frac{\varkappa}{2\alpha v}}\|u|L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A_{p,q}^s(\mathbb{R}^n))\|^2 \end{aligned} \tag{4.2.12}$$

with

$$\delta = (-\alpha\left(1+\frac{n}{r}\right) + \alpha g + a)v + 1 = \left(\alpha g - \frac{1}{v} - \alpha\lambda\right)v + 1 > 0 \tag{4.2.13}$$

since $\lambda < g$ and

$$\varkappa = \alpha - \frac{2}{v} - \frac{n\alpha}{r} - a)v + 1 = \left(-\frac{1}{v} - \frac{2n\alpha}{r} + \alpha\lambda\right)v + 1 > 0 \tag{4.2.14}$$

since $\frac{2n}{r} < \lambda$. Thus, T_{u_0} maps the unit ball U_T in $L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A_{p,q}^s(\mathbb{R}^n))$ into itself if T is sufficiently small.

As for the contraction consider $u, v \in U_T$. A similar calculation as above yields

$$\begin{aligned} &\|T_{u_0}u(\cdot,t) - T_{u_0}v(\cdot,t)|A_{p,q}^s(\mathbb{R}^n)\| \\ &\le c\,t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{n}{2r}-\frac{a}{\alpha}}\left(\int_0^t \tau^{av}\|Du^2(\cdot,\tau) - Dv^2(\cdot,\tau)|A_{p,q}^{s-\alpha(1+\frac{n}{r})}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v} \\ &\le c\,t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{n}{2r}-\frac{a}{\alpha}}\left(\int_0^t \tau^{av}\|u^2(\cdot,\tau) - v^2(\cdot,\tau)|A_{p,q}^{s-\frac{n}{r}}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}. \end{aligned} \tag{4.2.15}$$

Applying again Proposition 4.2.1 and Hölder's inequality this leads to

$$\begin{aligned} \|T_{u_0}u(\cdot,t) - T_{u_0}v(\cdot,t)|A_{p,q}^s(\mathbb{R}^n)\| &\le c\,t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{n}{2r}-\frac{a}{\alpha}} \\ &\times \left(\int_0^t \tau^{av}\|u(\cdot,\tau) - v(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{1/2\alpha v} \\ &\times \left(\int_0^t \tau^{av}\|u(\cdot,\tau) + v(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{1/2\alpha v}. \end{aligned} \tag{4.2.16}$$

Let temporarily $X_T^s = L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A_{p,q}^s(\mathbb{R}^n))$, then

$$\|T_{u_0}u - T_{u_0}v|X_T^s\| \le c\,T^{\frac{\varkappa}{2\alpha v}}\|u - v|X_T^s\| \tag{4.2.17}$$

with the same $\varkappa$ as (4.2.14). Hereby we estimated the second integral in (4.2.16) and used $u, v \in U_T$. If $T > 0$ is small enough, then $T_{u_0} : U_T \mapsto U_T$ is a contraction. Since we deal with Banach spaces we have shown that T_{u_0} has a unique fixed point in U_T which is a solution of (3.3.1).

Step 2. Similar to Theorem 3.3.5 we want to show that the solution gained in Step 1 is not only unique in the unit ball but in the whole space $X_T^s = L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))$. We will only emphazise which parts of Step 2 in Theorem 3.3.5 have to be adapted to the modified conditions. The idea consists again in iterating (4.2.10) - (4.2.13). We apply Proposition 3.2.3 now with $s-\alpha(1+\frac{n}{r})$ in place of s and $d=\alpha(1+\frac{n}{r})+\eta$, $\eta>0$ such that $d<2\big(\alpha-\frac{1}{v}\big)$, which is possible since $\frac{1}{\alpha v}\le\frac{1}{2}\big(1-\frac{n}{r}\big)$ and $r>n$. Then we obtain after the first iteration

$$\begin{aligned}\|T_{u_0}u(\cdot,t)|A^{s+\eta}_{p,q}(\mathbb{R}^n)\| &\le c\,t^{-\frac{\alpha(1+\frac{n}{r})+\eta-\alpha g}{2\alpha}}\|u_0|A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\|\\ &\quad + c\,t^{1-\frac{1}{\alpha v}-\frac{\alpha(1+\frac{n}{r})+\eta}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t\tau^{av}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}.\end{aligned}\tag{4.2.18}$$

Since for a solution u of (4.2.6), (4.2.7) it holds $T_{u_0}u=u$ we have for any $\varepsilon>0$ that $u_\varepsilon(x)=u(x,\varepsilon)\in A^{s+\eta}_{p,q}(\mathbb{R}^n)\hookrightarrow A^{s+\eta-\alpha(1+\frac{n}{r})+\alpha g}_{p,q}(\mathbb{R}^n)$. So we can choose new initial data u_ε with a sufficiently small $\varepsilon>0$. This yields after the k-th iteration step to

$$\begin{aligned}\|T_{u_\varepsilon}u|A^{s+k\eta}_{p,q}(\mathbb{R}^n)\| &\le c\,t^{-\frac{\alpha(1+\frac{n}{r})+\eta-\alpha g}{2\alpha}}\|u_\varepsilon|A^{s+(k-1)\eta-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\|\\ &\quad + c\,t^{1-\frac{1}{\alpha v}-\frac{\alpha(1+\frac{n}{r})+\eta}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t\tau^{av}\|Du^2(\cdot,\tau)|A^{s+(k-1)\eta-\alpha\left(1+\frac{n}{r}\right)}_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}\\ &\le c\,t^{-\frac{\alpha(1+\frac{n}{r})\eta-\alpha g}{2\alpha}}\|u_\varepsilon|A^{s+(k-1)\eta-\alpha\left(1-\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\|\\ &\quad + c\,t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{n}{2r}-\frac{\eta}{2\alpha}-\frac{a}{\alpha}}\left(\int_0^t\tau^{av}\|u(\cdot,\tau)|A^{s+(k-1)\eta}_{p,q}(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}.\end{aligned}\tag{4.2.19}$$

In the last step we applied Proposition 4.2.1 with $s+(k-1)\eta$ in place of s. Now one proceeds similar to Step 2 in Theorem 3.3.5 using the above estimates (4.2.18) and (4.2.19).

Step 3 and **Step 4** are proven analogously to Theorem 3.3.5 now with initial data u_0 belonging to the space $A^{s-\alpha(1+\frac{n}{r})+\alpha g}_{p,q}(\mathbb{R}^n)$ and the estimate

$$\|T_{u_0}u(\cdot,t)-T_{u_0}v(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\le c\,t^{\frac{1}{2}-\frac{1}{av}-\frac{n}{2r}-\frac{a}{\alpha}}\|u-v|X^s_{T_0}\|\|u+v|X^s_{T_0}\|\tag{4.2.20}$$

instead of (3.3.25). Further, we use Lemma 4.2.2 instead of Lemma 3.3.3.

Step 5. We show the continuity of the solution $u\in A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)$ up to $t=0$.

Analogously to Step 5 in Theorem 3.3.5 it holds

$$\begin{aligned}
&\|u(\cdot,t)-u_0|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)\| \\
&\leq c\,\|W_t^\alpha u_0-u_0|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)\|+\int_0^t\|W_{t-\tau}^\alpha Du^2(\cdot,\tau)|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)\|\mathrm{d}\tau.
\end{aligned} \tag{4.2.21}$$

Then the estimate of the first summand in $A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)$ works exactly the same as in (3.3.29) - (3.3.31). Regarding the second summand we obtain

$$\begin{aligned}
\int_0^t\|W_{t-\tau}^\alpha Du^2|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)\|\mathrm{d}\tau &\leq c\int_0^t(t-\tau)^{-\frac{\alpha g}{2\alpha}}\|Du^2(\cdot,\tau)|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)}(\mathbb{R}^n)\|\mathrm{d}\tau \\
&\leq c\int_0^t(t-\tau)^{-\frac{g}{2}}\|u(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^2\mathrm{d}\tau \\
&\leq c\,t^{1-\frac{1}{\alpha v}-\frac{g}{2}-\frac{a}{\alpha}}\left(\int_0^t\tau^{av}\|u(\cdot,\tau)|A_{p,q}^s(\mathbb{R}^n)\|^{2\alpha v}\mathrm{d}\tau\right)^{\frac{1}{\alpha v}}.
\end{aligned} \tag{4.2.22}$$

If $v<\infty$ we need $a=\alpha\left(1+\frac{n}{r}\right)-\frac{1}{v}-\alpha\lambda\leq\alpha-\frac{\alpha g}{2}-\frac{1}{v}$ to ensure that (4.2.22) tends to zero if t tends to zero. This leads to the restriction $\lambda\geq\frac{n}{r}+\frac{g}{2}$. If $v=\infty$ then $\lambda>\frac{n}{r}+\frac{g}{2}$ is required. $\square$

Remark 4.2.4. Note that from $r>n\geq 2$ and $\frac{1}{r}=\frac{1}{p}-\frac{s}{n}$ it follows on one hand

$$\frac{n}{r}=\frac{n}{p}-s<1\quad\text{if, and only if,}\quad\frac{n}{p}-1<s$$

and hence, $s>\frac{n}{p}-2\alpha+1$. Thus we deal with supercritical function spaces. On the other hand we have

$$1>\frac{n}{r}\geq\frac{2}{r}\quad\text{if, and only if,}\quad\frac{1}{r}<\frac{1}{2}.$$

Thus, the condition in Proposition 4.2.1 is satisfied.

We show that the strong solution of 4.0.1, 4.0.2 is also stable in the sense of Section 3.4 restricting ourselves again to the case $v=\infty$. In particular it satisfies (3.4.1) for a suitable δ such that (3.4.2) holds.

Theorem 4.2.5. *Let $n\geq 2$, $\alpha\in\mathbb{N}$, $1\leq p,q\leq\infty$ and $s>0$. Let $\frac{1}{r}=\frac{1}{p}-\frac{s}{n}$ and $\infty>r>n$ ($p<\infty$ for F- spaces, $q\leq r$ for B- spaces). Further, let a, g, λ as in Theorem 4.2.3 with $v=\infty$ whereas λ as in (4.2.8) and u_i, $i=1,2$ be solutions of (4.0.1), (4.0.2) obtained in Theorem 4.2.3 with initial data $u_0^i\in A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)$ in the corresponding time interval $(0,T_i)$. Then*

$$\|u_1(\cdot,t)-u_2(\cdot,t)|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)\|\leq c_0\|u_0^1-u_0^2|A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)\|+c_1\,t^{-\frac{g}{2}-\frac{a}{\alpha}+1}$$

for all $0 < t < T := \min(T_1, T_2)$. *The constants* $c_0 > 0$ *and* $c_1 > 0$ *are independent of the initial data and* t. *In particular the solution* u *is locally stable. If additionally* $\max(p, q) < \infty$ *then the Cauchy problem* (4.0.1), (4.0.2) *is well-posed.*

Proof. Let u_1, u_2 be two solutions of (3.3.1) with corresponding initial data u_0^1, u_0^2. Then it holds

$$\begin{aligned}
&\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\| \le \|W^\alpha_t(u_0^1 - u_0^2)|A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&+ \int_0^t \|W^\alpha_{t-\tau}(Du_1^2 - Du_2^2)(\cdot,\tau)|A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau. \qquad (4.2.23)
\end{aligned}$$

Now we apply Theorem 3.2.2 with $d = 0$ to the first summand and with $d = \alpha g$ and $s - \alpha\left(1 + \frac{n}{r}\right)$ in place of s to the second summand. This yields

$$\begin{aligned}
&\|u_1(\cdot,t) - u_2(\cdot,t)|A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\| \le c_0\,\|u_0^1 - u_0^2|A^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&+ c\int_0^t (t-\tau)^{-g/2}\|D(u_1^2 - u_2^2)(\cdot,\tau)|A^{s-\alpha\left(1+\frac{n}{r}\right)}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau.
\end{aligned}$$

By means of Minkowski's inequality we can split the norm in the second summand and estimate the terms with u_1 and u_2 separately. Let u be either u_1 or u_2. Then we obtain, using similar arguments as in the proof of Theorem 4.2.3,

$$\begin{aligned}
&\int_0^t (t-\tau)^{-g/2}\|Du^2(\cdot,\tau)|A^{s-\alpha\left(1+\frac{n}{r}\right)}_{p,q}(\mathbb{R}^n)\|\,\mathrm{d}\tau \lesssim \int_0^t (t-\tau)^{-g/2}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^2\,\mathrm{d}\tau \\
&= \int_0^t (t-\tau)^{-g/2}\tau^{-a/\alpha}(\tau^{a/2\alpha}\,\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|)^2\,\mathrm{d}\tau \\
&\le \int_0^t (t-\tau)^{-g/2}\tau^{-a/\alpha}\,\mathrm{d}\tau \le c_1 t^{-g/2-a/\alpha+1}, \quad 0 < t < T \qquad (4.2.24)
\end{aligned}$$

because of $\|u|L_\infty((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n))\| \le 1$. The exponent is strictly positive since

$$\frac{g}{2} + \frac{a}{\alpha} - 1 = \frac{g}{2} - 1 + \frac{1}{\alpha}\left(\alpha\left(1 + \frac{n}{r}\right) - \alpha\lambda\right) < 0$$

due to $\lambda > \frac{g}{2} + \frac{n}{r}$. Hence, (4.2.24) tends to zero if t tends to zero. That means one can choose for any $\varepsilon > 0$ a suitable $\delta > 0$ and an appropriate $T > 0$ such that (3.4.1) follows from (3.4.2) for any $0 < t < T$. In particular the solution of (4.0.1), (4.0.2) obtained in Theorem 4.2.3 is locally stable and thus the problem well-posed if $\max(p, q) < \infty$. □

4.3 The limiting case $\mathbf{s = \frac{n}{p}}$

We investigate now the limiting case $s = \frac{n}{p} > 0$ when the spaces $A^s_{p,q}(\mathbb{R}^n)$ are not multiplication algebras. That means in case of F-spaces if $1 < p < \infty$, $1 \le q \le \infty$ and in case of B-spaces if $1 \le p \le \infty$, $1 < q \le \infty$. The choice $s = \frac{n}{p}$ causes in Theorem 1.4.4 $\frac{1}{r} = \frac{1}{p} - \frac{s}{n} = 0$ such that the condition in Proposition 4.2.1 is not satisfied. That is the reason why we have to apply Theorem 1.4.5 to obtain an appropriate estimate for the term Du^2. After these preliminary remarks the following embeddings holds.

Proposition 4.3.1. *Let $n \ge 2$, $1 \le p < r < \infty$, $1 \le q \le \infty$ ($1 < p$ for F-spaces, $1 < q$ for B-spaces) and let $\frac{1}{p_r} = \frac{1}{p} + \frac{1}{r}$. Then*

$$A^{n/p}_{p,q}(\mathbb{R}^n) \cdot A^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{n/p}_{p_r,q}(\mathbb{R}^n).$$

Furthermore

$$\|D(uv)|A^{n/p-1-n/r}_{p,q}(\mathbb{R}^n)\| \le c\,\|uv|A^{n/p-n/r}_{p,q}(\mathbb{R}^n)\| \le c\,\|uv|A^{n/p}_{p_r,q}(\mathbb{R}^n)\| \tag{4.3.1}$$

$$\le c\,\|u|A^{n/p}_{p,q}(\mathbb{R}^n)\|\|v|A^{n/p}_{p,q}(\mathbb{R}^n)\|. \tag{4.3.2}$$

Proof. We apply Theorems 1.4.5 and 1.4.6 with p in place of p_1, p_r in place of p, $s = n/p$ and $s_1 = n/p - n/r$. □

The analogue of Lemma 4.2.2 reads as follows.

Lemma 4.3.2. Let $n \in \mathbb{N}$, $n \ge 2$, $\alpha \in \mathbb{N}$, $1 \le p < \infty$, $1 \le q \le \infty$ ($1 < p$ for F-spaces, $1 < q$ for B-spaces) and $\max(n,p) < r < \infty$. Let

$$\frac{2n}{r} < \lambda < g \le 1 + \frac{n}{r}, \quad 0 \le \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{r}\right), \quad a = \alpha\left(1 + \frac{n}{r}\right) - \frac{1}{v} - \alpha\lambda \tag{4.3.3}$$

and let $u_0 \in A^{\frac{n}{p}-\alpha\left(1+\frac{n}{r}\right)+\alpha g}_{p,q}(\mathbb{R}^n)$. Further assume that $u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^{\frac{n}{p}}_{p,q}(\mathbb{R}^n))$ is a fixed point of T_{u_0} as defined in (3.0.2). Then it holds

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2$$

in the sense of $D'(\mathbb{R}^n \times (0,T))$.

Proof. The proof is the same as that of Lemma 4.2.2 now with $s = \frac{n}{p} > 0$ using again $r > n$ and Proposition 4.3.1 instead of Proposition 4.2.1 with $u^2(\cdot,t) \in A^{n/p}_{p_r,q}(\mathbb{R}^n)$ for fixed $t \in (0,T)$. □

Thus we obtain the following corresponding existence and uniqueness theorem.

Theorem 4.3.3. *Let $n \in \mathbb{N}$, $n \ge 2$, $\alpha \in \mathbb{N}$, $1 \le p < \infty$, $1 \le q \le \infty$ ($1 < p$ for F-spaces, $1 < q$ for B-spaces) and* $\max(n,p) < r < \infty$.

(i) *Let*

$$\frac{2n}{r} < \lambda < g \leq 1 + \frac{n}{r}, \quad 0 \leq \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{r}\right), \quad a = \alpha\left(1 + \frac{n}{r}\right) - \frac{1}{v} - \alpha\lambda \quad (4.3.4)$$

and let $u_0 \in A_{p,q}^{\frac{n}{p}-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)$ *for the initial data. Then there exists a number* $T > 0$ *such that*

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= 0, \qquad \text{in } \mathbb{R}^n \times (0,T), && (4.3.5)\\ u(x,0) &= u_0(x), \quad \text{in } \mathbb{R}^n && (4.3.6) \end{aligned}$$

has a unique solution

$$u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A_{p,q}^{\frac{n}{p}}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T)).$$

(ii) *If, in addition,* $q < \infty$ *and*

$$\frac{g}{2} + \frac{n}{r} \leq \lambda < g \leq 1 + \frac{n}{r} \text{ if } v < \infty \text{ and } \frac{g}{2} + \frac{n}{r} < \lambda < g \leq 1 + \frac{n}{r} \text{ if } v = \infty$$

then the above solution is strong, that means $u \in C([0,T), A_{p,q}^{\frac{n}{p}-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n))$.

(iii) *Under the additional assumptions on* λ *imposed in Part (**ii**) the solution* u *obtained in Part (**i**) is locally stable and hence the problem well-posed if additionally* $q < \infty$.

Proof. The proof coincides exactly with those of Theorems 4.2.3 and 4.2.5 replacing s by $\frac{n}{p}$ and using Proposition 4.3.1 instead of Proposition 4.2.1. □

4.4 The case $F_{p,2}^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, $2 \leq n < p < \infty$

Of special interest is the case $F_{p,2}^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, $1 < p < \infty$. We consider only supercritical function spaces $L_p(\mathbb{R}^n)$, i.e. spaces with $2 \leq n < p < \infty$. The following Proposition deals with the mapping properties of Du^2 under these conditions.

Proposition 4.4.1. *Let* $2 \leq n < p < \infty$. *Then it holds*

$$\|D(uv)|F_{p,2}^{-1-n/p}(\mathbb{R}^n)\| \leq c\,\|u|L_p(\mathbb{R}^n)\|\|v|L_p(\mathbb{R}^n)\|, \quad u, v \in L_p(\mathbb{R}^n).$$

Proof. We apply Hölder's inequality

$$L_p(\mathbb{R}^n) \cdot L_p(\mathbb{R}^n) \hookrightarrow L_{p/2}(\mathbb{R}^n)$$

and Theorem 1.4.6 with $p/2$ in place of p, p in place of p_1, $s = 0$, $s_1 = -\frac{n}{p}$ and $q = 2$. Then we have

$$L_{p/2}(\mathbb{R}^n) = F^0_{p/2,2}(\mathbb{R}^n) \hookrightarrow F^{-n/p}_{p,2}(\mathbb{R}^n)$$

and hence

$$\begin{aligned} \|D(uv)|F^{-1-n/p}_{p,2}(\mathbb{R}^n)\| &\le c\,\|uv|F^{-n/p}_{p,2}(\mathbb{R}^n)\| \le c\,\|uv|F^0_{p/2,2}\| && (4.4.1)\\ &\le c\,\|u|L_p(\mathbb{R}^n)\|\|v|L_p(\mathbb{R}^n)\|. && (4.4.2)\end{aligned}$$

□

We formulate very shortly the analogue of Lemma 4.2.2.

Lemma 4.4.2. Let $\alpha \in \mathbb{N}$ and $n \in \mathbb{N}$ with $2 \le n < p < \infty$.

$$\frac{2n}{p} < \lambda < g \le 1 + \frac{n}{p}, \quad 0 \le \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{p}\right), \quad a = \alpha\left(1 + \frac{n}{p}\right) - \frac{1}{v} - \alpha\lambda \qquad (4.4.3)$$

and $u_0 \in F^{-\alpha\left(1+\frac{n}{p}\right)+\alpha g}_{p,2}(\mathbb{R}^n)$. Further assume that $u \in L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, L_p(\mathbb{R}^n))$ is a fixed point of T_{u_0} as defined in (3.0.2). Then it holds

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2$$

in the sense of $D'(\mathbb{R}^n \times (0,T))$.

Proof. Proceeding similarly to Lemma 4.2.2 we check that $s_0 = -\alpha\left(1 + \frac{n}{p}\right) + \alpha g > -2\alpha$. But this is satisfied since $n < p$. Hence, W^α belongs to $L^{\text{loc}}_1(\mathbb{R}^{n+1})$.
From Proposition 4.4.1 we know that $u^2(\cdot,t) \in L_{p/2}(\mathbb{R}^n)$, $2 < p < \infty$ for fixed $0 < t < T$. By Lemma 2.3.5 and Lemma 3.3.2 it follows with αv in place of v and $\frac{a}{2\alpha}$ in place of b that $U^2 \in L^{\text{loc}}_1(\mathbb{R}^{n+1})$ if $u \in L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n))$. Moreover, $U \in L^{\text{loc}}_1(\mathbb{R}^{n+1})$.
Thus, using Theorem 3.2.2 and the fact that $W^\alpha_{t-\tau}u^2(\cdot,\tau) \in C^\infty(\mathbb{R}^n)$ we obtain for any compact subset $\mathcal{K} \subset \mathbb{R}^{n+1}$ and some $\frac{n}{p} < d < 2\left(\alpha - \frac{1}{v}\right)$

$$\begin{aligned} \int_{\mathcal{K}} |H(x,t)|\,\mathrm{d}(x,t) &\lesssim \int_0^R \int_{|x|<R} \int_0^t |W^\alpha_{t-\tau}u^2(x,\tau)|\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \int_0^R \int_0^t \|W^\alpha_{t-\tau}u^2(\cdot,\tau)|L_p(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t \\ &\lesssim \int_0^R \int_0^t \|W^\alpha_{t-\tau}u^2(\cdot,\tau)|F^{-\frac{n}{p}+d}_{p,2}(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t \\ &\lesssim \int_0^R \int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\|u^2(\cdot,\tau)|F^{-\frac{n}{p}}_{p,2}(\mathbb{R}^n)\|\,\mathrm{d}\tau\,\mathrm{d}t \\ &\lesssim \int_0^R \int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\tau^{-\frac{a}{\alpha}}\tau^{\frac{a}{\alpha}}\|u(\cdot,\tau)|L_p(\mathbb{R}^n)\|^2\,\mathrm{d}\tau\,\mathrm{d}t && (4.4.4) \\ &\lesssim R^{2-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\|u|L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, L_p(\mathbb{R}^n))\|^2. \end{aligned}$$

In (4.4.4) we used Proposition 4.4.1. The choice of d is possible since

$$0 < \frac{n}{p} < 1 - \frac{2}{\alpha v} \leq \alpha - \frac{2}{v} < 2\left(\alpha - \frac{1}{v}\right).$$

Defining $K = D(G^\alpha * U^2) = G^\alpha * DU^2$ the remaining steps coincide exactly with those of Lemma 3.3.3. Hence,

$$(\partial_t + (-\Delta_x)^\alpha)u = Du^2 \quad \text{in} \quad D'(\mathbb{R}^n \times (0,T)).$$

□

Using these embeddings we obtain the analogous result for solutions of 4.0.1, 4.0.2.

Theorem 4.4.3. *Let $\alpha \in \mathbb{N}$ and $2 \leq n < p < \infty$.*

(i) *Let*

$$\frac{2n}{p} < \lambda < g \leq 1 + \frac{n}{p}, \quad 0 \leq \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{p}\right), \quad a = \alpha\left(1 + \frac{n}{p}\right) - \frac{1}{v} - \alpha\lambda \tag{4.4.5}$$

and $u_0 \in F_{p,2}^{-\alpha\left(1+\frac{n}{p}\right)+\alpha g}(\mathbb{R}^n)$ for the initial data. Then there exists a number $T > 0$ such that

$$\begin{aligned} \partial_t u(x,t) + (-\Delta_x)^\alpha u(x,t) - Du^2(x,t) &= 0, && \text{in } \mathbb{R}^n \times (0,T), && (4.4.6)\\ u(x,0) &= u_0(x), && \text{in } \mathbb{R}^n, && (4.4.7) \end{aligned}$$

has a unique solution

$$u \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, L_p(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T)).$$

(ii) *If, in addition,*

$$\frac{g}{2} + \frac{n}{p} \leq \lambda < g \leq 1 + \frac{n}{p} \text{ if } v < \infty \text{ and } \frac{g}{2} + \frac{n}{p} < \lambda < g \leq 1 + \frac{n}{p} \text{ if } v = \infty$$

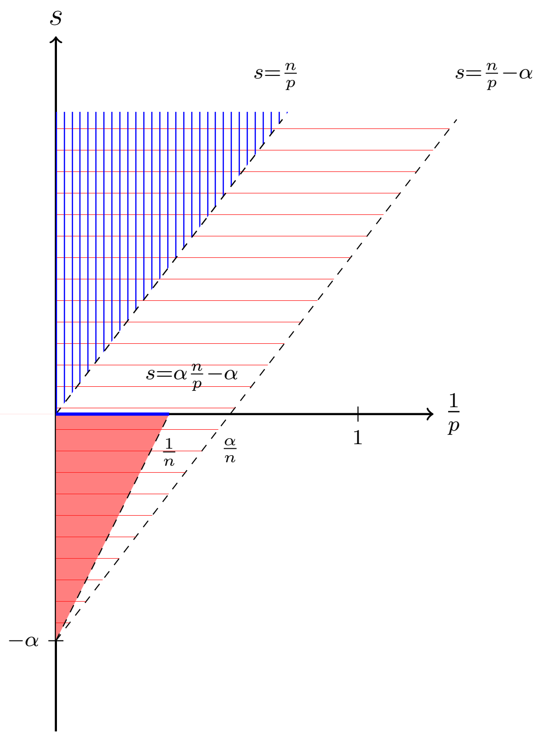
then the above solution is strong, that means $u \in C([0,T), F_{p,2}^{-\alpha\left(1+\frac{n}{p}\right)+\alpha g}(\mathbb{R}^n))$.

(iii) *Under the additional assumption on the parameter λ imposed in Part **(ii)** the solution u obtained in Part **(i)** is locally stable and hence the problem* (4.0.1), (4.0.2) *well-posed in the setting of this theorem.*

Proof. One can follow the estimates in Theorem 4.2.3 and Theorem 4.2.5 with the following adjustments. To prove Part (**i**) one has to choose $d = \alpha\left(1+\frac{n}{p}\right)$ and $s = 0$ in Steps 1 - 4 of Theorem 4.2.3. As for Part (**ii**) Step 5 has to be reproduced with $d = \alpha g$ and $s = -\alpha\left(1+\frac{n}{p}\right)$. Concerning the proof of the stability of the strong solution obtained in Parts (**i**) and (**ii**) the first summand in (4.2.23) has to be estimated with $d = 0$, $s = -\alpha\left(1+\frac{n}{p}\right) + \alpha g$ and the second summand again with $d = \alpha g$ and $s = -\alpha\left(1+\frac{n}{p}\right)$. □

4.5 Comparison of the solution spaces

We analyze how the solution spaces change in dependence of different conditions on the smoothness of the solution. We restrict ourselves to the case when $u(\cdot,t) \in H_p^s(\mathbb{R}^n)$, $s > n/p$, for fixed $t > 0$ compared with the case $u(\cdot,t) \in L_p(\mathbb{R}^n)$, $2 \le n < p < \infty$. As for the remaining spaces one can proceed similarly. In the first case we can choose by Theorem 3.3.5 initial data $u_0 \in H_p^{s-\alpha+\alpha g}(\mathbb{R}^n)$ with $\alpha \in \mathbb{N}$, $1 < p < \infty$ and $0 < g \le 1$ shaded with red stripes in the picture below. That means they have a smoothness $s_0 > s - \alpha > n/p - \alpha$. The corresponding solution u belongs to $H_p^s(\mathbb{R}^n)$ with $s > n/p$ which is represented by the blue shaded area.



To receive solutions in $L_p(\mathbb{R}^n)$ with $2 \le n < p < \infty$ we can start with $u_0 \in H_p^{-\alpha(1+\frac{n}{p})+\alpha g}(\mathbb{R}^n)$

with $2n/p < g \leq 1 + n/p$ represented by the red part. Hence, the possible smoothness of the initial data can almost be $-\alpha(1 - n/p)$. The blue line marks the space $L_p(\mathbb{R}^n)$ with $p > n$.
Concerning the dependence on t the condition $s > n/p$ results in solution spaces $L_{2\alpha v}((0,T), \frac{a}{2\alpha}, H_p^s(\mathbb{R}^n))$ with $a \in (-\alpha/2, \alpha)$ and $1/v \in [0, \alpha/2)$. In the $L_p(\mathbb{R}^n)$- case we have solution spaces with $a \in (-\alpha/2(1 - n/p), \alpha(1 - n/p))$ and $1/v \in [0, \alpha/2(1 - n/p))$. Hence, the range of the parameters a and v in the $L_p(\mathbb{R}^n)$- case is completely included in the range of parameters when $s > n/p$.
On the other hand, in particular situations the lowering of the smoothness results in a higher integrability as the following example shows. Let $u_0 \in H_p^{-\frac{\alpha n}{p}}(\mathbb{R}^n)$, i.e. we choose $g = 1$. This is possible if $p > 2n$. For λ it follows $2n/p < \lambda < 1$ and hence, $1 - \lambda > 0$. Concerning the solution space $L_{2\alpha v}((0,T), \frac{a}{2\alpha}, H_p^s(\mathbb{R}^n))$ it holds for the power of the weight $t^{a/2\alpha}$

$$\frac{a}{\alpha} = 1 + \frac{n}{p} - \frac{1}{\alpha v} - \lambda > \frac{n}{p} - \frac{1}{\alpha v}$$

where we omitted the factor 1/2 for a better readability. Let $s > n/p$ and assume that $u_0 \in H_p^{-\frac{\alpha n}{p}}(\mathbb{R}^n)$. That means

$$s - \alpha(1 - g) = -\frac{\alpha n}{p}, \quad \text{if, and only if,} \quad 1 - g = \frac{s}{\alpha} + \frac{n}{p}.$$

Because of $0 < \lambda < g$ and $a = \alpha - 1/v - \alpha\lambda$ we obtain with the same α, v as above

$$\frac{s}{\alpha} + \frac{n}{p} < 1 - \lambda = \frac{a}{\alpha} + \frac{1}{\alpha v}.$$

Thus,

$$\frac{a}{\alpha} > \frac{s}{\alpha} + \frac{n}{p} - \frac{1}{\alpha v} > \frac{n}{\alpha p} + \frac{n}{p} - \frac{1}{\alpha v} > \frac{n}{p} - \frac{1}{\alpha v}.$$

We summarize the outcome. Starting in both cases with the same initial data results in the $L_p(\mathbb{R}^n)$- case in a solution with a better integrability, more precisely in a lower power of $t^{a/2\alpha}$ which leads to possible larger values of the weight in a neighborhood of zero. Hence, it makes sense to study solutions with different conditions on the smoothness separately.

5 Generalized Navier-Stokes equations

In this part we will apply the results for the generalized nonlinear heat equation as presented in the previous chapters. As indicated in Section 2.4 we are interested in the following problems. Let $\boldsymbol{u}(x,t) = (u^1(x,t), \cdots, u^n(x,t))$ be a velocity field and $P(x,t)$ a scalar pressure, $x \in \mathbb{R}^n$, $t \in (0,T)$. Then the generalized Navier-Stokes equations are given by

$$\begin{array}{rcll} (\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u} + (\boldsymbol{u},\nabla)\boldsymbol{u} + \nabla P & = & 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} \boldsymbol{u} & = & 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot,0) & = & \boldsymbol{u}_0 & \text{in } \mathbb{R}^n. \end{array} \tag{5.0.1}$$

In particular we deal with its reformulation

$$\begin{array}{rcll} (\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u} + \mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) & = & 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \boldsymbol{u}(\cdot,0) & = & \boldsymbol{u}_0 & \text{in } \mathbb{R}^n \end{array} \tag{5.0.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $0 < T \leq \infty$ and $\mathbb{P}$ denotes the Leray projector as given in (2.4.6). Furthermore, ∇, div and $\otimes$ are defined as in Section 2.4.
Our method is as follows. First we point out the conditions under which (5.0.1) and (5.0.2) are equivalent. It turns out that this holds true under the additional assumption $\operatorname{div}\boldsymbol{u}_0 = 0$. In the following course of the chapter we stick on (5.0.2). As already done in Chapters 3 and 4 for the generalized nonlinear heat equation we distinguish between the following cases. First we ask for solutions $\boldsymbol{u}$ of (5.0.2) which belong for fixed $t \in (0,T)$ to some spaces $A^s_{p,q}(\mathbb{R}^n)_n$ which are multiplication algebras. Then, concerning the spaces $A^s_{p,q}(\mathbb{R}^n)_n$, we concentrate on the strip $\frac{n}{p} - 1 < s < \frac{n}{p}$. Further, we consider the limiting case $s = \frac{n}{p}$ when $A^s_{p,q}(\mathbb{R}^n)_n$ is not a multiplication algebra and the special case $\boldsymbol{u} \in F^0_{p,2}(\mathbb{R}^n)_n$ in the framework of supercritical function spaces. The dependence of the solution on time t will be described in terms of weighted vector-valued Lebesgue spaces $L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)_n)$. For this purpose we consider the vector-valued version of (3.0.2), i.e.

$$T_{\boldsymbol{u}_0}\boldsymbol{u}(x,t) = W^\alpha_t \boldsymbol{u}_0(x) - \int_0^t W^\alpha_{t-\tau}(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T. \tag{5.0.3}$$

We justify that a fixed point of the operator $T_{\boldsymbol{u}_0}$ is also a solution of

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u} + \mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

in the sense of $D'(\mathbb{R}^n \times (0,T))$, cf. Lemmas 3.3.3, 4.2.2, 4.3.2 and 4.4.2. To this end we show that (3.0.1) is the scalar case of (5.0.2).

5.1 Reformulation

From now on we assume that $\boldsymbol{u}$ belongs for fixed $t \in (0,T)$ to some space $A^s_{p,q}(\mathbb{R}^n)_n$ such that (2.4.10) is fulfilled. Further, we assume that $A^s_{p,q}(\mathbb{R}^n)_n$ possesses suitable multiplication properties as indicated in the introduction of this chapter. This implies in particular $s > 0$, except in the special case $F^0_{p,2}(\mathbb{R}^n)_n = L_p(\mathbb{R}^n)$, $2 \le n < p < \infty$, and thus $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$, $1 < p < \infty$. Consequently, we deal with functions such that a pointwise multiplication is always well defined. Moreover, the assertions obtained in Proposition 2.4.1 can be extended to these spaces and Proposition 2.4.2 holds.

We justify that one can reduce (5.0.1) to (5.0.2) under the above described conditions. Let $\boldsymbol{u}$ be a fixed point of $T_{\boldsymbol{u}_0}$ as given in (5.0.3). Then it can be represented by

$$\boldsymbol{u}(x,t) = W^\alpha_t \boldsymbol{u}_0(x) - \int_0^t W^\alpha_{t-\tau}(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T. \tag{5.1.1}$$

Further, assume $\operatorname{div}\boldsymbol{u}_0 = 0$, or equivalently $\sum_{j=1}^n \xi_j\, \widehat{u^j_0}(\xi) = 0$. We show that then also $\operatorname{div}\boldsymbol{u} = 0$. It holds that

$$(\operatorname{div}\boldsymbol{u})^\wedge(\xi,t) = i\sum_{j=1}^n \xi_j\, \widehat{u^j}(\xi) \tag{5.1.2}$$

$$= ie^{-t|\xi|^{2\alpha}} \sum_{j=1}^n \xi_j\, \widehat{u^j_0}(\xi) - i\int_0^t e^{-(t-\tau)|\xi|^{2\alpha}} \sum_{j=1}^n \xi_j (\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^{\wedge j}(\xi,\tau)\mathrm{d}\tau. \tag{5.1.3}$$

By our assumption the first summand equals zero. Since $\mathbb{P}$ is the projection of $A^s_{p,q}(\mathbb{R}^n)$ onto div $A^s_{p,q}(\mathbb{R}^n)$, cf. Proposition 2.4.1 and the above remarks, it follows

$$\sum_{j=1}^n \xi_j (\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^{\wedge j}(\xi,\tau) = (\operatorname{div}\mathbb{P}(\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})))^\wedge(\xi,\tau) = 0 \tag{5.1.4}$$

and

$$(\operatorname{div}\boldsymbol{u})^\wedge(\xi,t) = 0. \tag{5.1.5}$$

Hence, the property of being divergence-free is inherited from $\boldsymbol{u}_0$ to $\boldsymbol{u}$.

Proposition 5.1.1. *Let either be $\boldsymbol{u}(\cdot,t) \in A^s_{p,q}(\mathbb{R}^n)_n$ with $1 < p < \infty$, $0 < q \le \infty$, $s > 0$ or $\boldsymbol{u}(\cdot,t) \in L_p(\mathbb{R}^n)$ with $2 \le n < p < \infty$, $0 < t < T \le \infty$.*

(i) *Let* $\boldsymbol{u}$ *be a solution of* (5.0.2)*,* $\operatorname{div}\boldsymbol{u}_0 = 0$ *and* $P = \sum_{l,j=1}^{n} R_j\, R_l(u^l u^j)$*. Then* $\boldsymbol{u}$ *is a solution of* (5.0.1)*, too.*

(ii) *Let* $\boldsymbol{u}$ *be a solution of* (5.0.1) *and assume that* P *belongs to a space that covers* (2.4.9)*. Then* $\boldsymbol{u}$ *is a solution of* (5.0.2)*, too.*

Proof. **Step 1.** We show (i). According to (5.1.1) - (5.1.5) it follows, that $\operatorname{div}\boldsymbol{u} = 0$ with the consequence, that we can write $(\boldsymbol{u},\nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})$. Then $\boldsymbol{u}$ is a solution of (5.0.1) and the unknown pressure is determined uniquely up to a constant by

$$P = \sum_{l,j=1}^{n} R_j\, R_l(u^l u^j) \quad \text{if, and only if,} \quad \nabla P = -\mathbb{Q}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}), \tag{5.1.6}$$

cf. [37, pp. 197, 198].

Step 2. To prove the converse we show first that R_k and partial derivatives can be interchanged. The exchange of R_k and ∂_t is immediately clear. For ∂_j we obtain

$$R_k\partial_j u^l(x,t) = i\left(\frac{\xi_k}{|\xi|}(\partial_j u^l)^\wedge\right)^\vee(x,t) = -\left(\frac{\xi_k\xi_j}{|\xi|}\widehat{u^l}\right)^\vee(x,t) \tag{5.1.7}$$

and

$$\partial_j R_k u^l(x,t) = i\partial_j\left(\frac{\xi_k}{|\xi|}\widehat{u^l}\right)^\vee(x,t) = -\left(\frac{\xi_k\xi_j}{|\xi|}\widehat{u^l}\right)^\vee(x,t), \tag{5.1.8}$$

$j,k,l = 1,\dots,n$. Hence,

$$\partial_j\mathbb{P} = \mathbb{P}\partial_j. \tag{5.1.9}$$

Let $\boldsymbol{u}$, P be a solution of (5.0.1). Then $\boldsymbol{u}$, P is also a solution of

$$\begin{aligned} \partial_t\mathbf{u} + (-\Delta_x)^\alpha\mathbf{u} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla P &= 0 \quad \text{in } \mathbb{R}^n\times(0,T),\\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0 \quad \text{in } \mathbb{R}^n \end{aligned} \tag{5.1.10}$$

and of

$$\begin{aligned} \mathbb{P}(\partial_t\mathbf{u} + (-\Delta_x)^\alpha\mathbf{u} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla P) &= 0 \quad \text{in } \mathbb{R}^n\times(0,T),\\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{5.1.11}$$

Now we use (5.1.9), $\mathbb{P}\nabla P = 0$ and $\mathbb{P}\boldsymbol{u} = \boldsymbol{u}$ due to $\operatorname{div}\boldsymbol{u} = 0$ and obtain the desired result. □

5.2 Main results

Before we show the existence and uniqueness of a strong solution for the reformulated generalized Navier-Stokes equations, i.e. for (5.0.2) we justify that under the assumptions of the theorem below a fixed point of (5.0.3) is also a solution of

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u} = -\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}).$$

Lemma 5.2.1. Let $n\in\mathbb{N}$, $n\geq 2$, $\alpha\in\mathbb{N}$, $1<p<\infty$, $1\leq q\leq\infty$ and s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra. Let

$$0<\lambda<g\leq 1,\ \ \frac{2}{\alpha}<v\leq\infty,\ \ a=\alpha-\frac{1}{v}-\alpha\lambda \tag{5.2.1}$$

and let $\boldsymbol{u}_0\in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)_n$. Further assume that $\boldsymbol{u}\in L_{2\alpha v}((0,T),\frac{a}{2\alpha},A^s_{p,q}(\mathbb{R}^n)_n)$ is a fixed point of $T_{\boldsymbol{u}_0}$ as defined in (5.0.3). Then it holds

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u} = -\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})$$

in the sense of $D'(\mathbb{R}^n\times(0,T))_n$.

Proof. **Step 1.** Since $\boldsymbol{u}$ is a fixed point of $T_{\boldsymbol{u}_0}$ it can be written as

$$\boldsymbol{u}(x,t)=W^\alpha_t\boldsymbol{u}_0(x)-\int_0^t W^\alpha_{t-\tau}(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))(x,\tau)\mathrm{d}\tau,\quad x\in\mathbb{R}^n,\ 0<t<T. \tag{5.2.2}$$

We consider the first summand. Let $\boldsymbol{W}^\alpha$ be the vector-valued version of (3.3.2), i.e.

$$\boldsymbol{W}^\alpha(x,t)=\begin{cases}W^\alpha_t\,\boldsymbol{u}_0(x), & x\in\mathbb{R}^n,\ t\in(0,T),\\ 0, & x\in\mathbb{R}^n,\ t\in\mathbb{R}\setminus(0,T).\end{cases} \tag{5.2.3}$$

To show that $\boldsymbol{W}^\alpha\in L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})_n$ let $\mathcal{K}\subset\mathbb{R}^{n+1}$ be compact. Then it holds

$$\begin{aligned}\int_{\mathcal{K}}|\boldsymbol{W}^\alpha(x,t)|\mathrm{d}(x,t)&\leq\int_0^R\int_{|x|\leq R}\sum_{k=1}^n|W^\alpha_t\,\boldsymbol{u}^k_0(x)|\mathrm{d}x\,\mathrm{d}t\\&=\sum_{k=1}^n\int_0^R\int_{|x|\leq R}|W^\alpha_t\,\boldsymbol{u}^k_0(x)|\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $s>-2\alpha$ we know from Lemma 3.3.1 that each component belongs to $L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})$ and hence $\boldsymbol{W}^\alpha\in L^{\mathrm{loc}}_1(\mathbb{R}^{n+1})_n$.

Step 2. We turn our attention to the second summand. Let $\boldsymbol{U}$ be the extension of $\boldsymbol{u}$ by zero to $\mathbb{R}^n \times (\mathbb{R} \setminus (0,T))$. Our first aim is to show that $\boldsymbol{U} \otimes \boldsymbol{U} \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})_{n\times n}$ if $\boldsymbol{u} \in L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)_n)$. According to Definition 1.2.25 we have

$$\begin{aligned}
&\|\boldsymbol{u}\otimes\boldsymbol{u}|L_{\alpha v}((0,T),\, \frac{a}{\alpha},\, A^s_{p,q}(\mathbb{R}^n)_{n\times n})\| \\
&= \left(\int_0^T t^{av}\left(\sum_{i,k=1}^n \|(u^i u^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{\alpha v} \mathrm{d}t\right)^{1/\alpha v} \\
&\lesssim \left(\int_0^T t^{av}\left(\sum_{i,k=1}^n \|u^i(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\|u^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{\alpha v} \mathrm{d}t\right)^{1/\alpha v} \\
&= \left(\int_0^T t^{av}\left(\sum_{k=1}^n \|u^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{2\alpha v} \mathrm{d}t\right)^{1/\alpha v} \\
&= \|\boldsymbol{u}|L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)_n)\|^2.
\end{aligned}$$

It follows from Lemma 2.3.5 that each component of $\boldsymbol{U}\otimes\boldsymbol{U}$ and each component of $\boldsymbol{U}$ belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})$, respectively, since $a < \alpha - \frac{1}{v}$ and $1 < v \le \infty$. Thus we have $\boldsymbol{U} \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})_n$ and $\boldsymbol{U}\otimes\boldsymbol{U} \in L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})_{n\times n}$.

Step 3. Now we verify that the convolution of G^α and $\mathbb{P}(\boldsymbol{U}\otimes\boldsymbol{U})$ exists and belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^{n+1})_{n\times n}$ whereas we apply the Leray projector column by column to $\boldsymbol{U}\otimes\boldsymbol{U}$. We define the analogue to H in Lemma 3.3.2 as follows

$$\boldsymbol{H}(x,t) = \begin{cases} \int\limits_0^t \int_{\mathbb{R}^n} G^\alpha(x-y,t-\tau)\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau, & x\in\mathbb{R}^n,\ 0<t<T \\ \int\limits_0^T \int_{\mathbb{R}^n} G^\alpha(x-y,t-\tau)\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau, & x\in\mathbb{R}^n,\ t\ge T \\ 0, & x\in\mathbb{R}^n,\ t\le 0. \end{cases}$$

The convolution with G^α has to be understood element by element. We consider only the case $0<t<T$ and reduce the proof to that of Lemma 3.3.2. We have

$$|\boldsymbol{H}(x,t)| = \sum_{k,l=1}^n |H_{k,l}(x,t)| \le \sum_{k,l=1}^n \int_0^t |W^\alpha_{t-\tau}[\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k(x,\tau)|\mathrm{d}\tau.$$

Then we obtain for any compact subset $\mathcal{K}\subset\mathbb{R}^{n+1}$ and some d with $0\le d< 2\left(\alpha-\frac{1}{v}\right)$ similar to Lemma 3.3.2

$$\begin{aligned}
\int_{\mathcal{K}} |\boldsymbol{H}(x,t)|\mathrm{d}(x,t) &\lesssim \int_0^R\int_0^t \sum_{k,l=1}^n \|W^\alpha_{t-\tau}[\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k(\cdot,\tau)|A^{s+d}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim \int_0^R\int_0^t \sum_{k,l=1}^n (t-\tau)^{-\frac{d}{2\alpha}}\|[\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau\,\mathrm{d}t. \qquad (5.2.4)
\end{aligned}$$

For the norm in (5.2.4) it holds

$$
\begin{aligned}
&\|[\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \|(u^l u^k)(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\| + \|R_k\sum_{j=1}^n R_j(u^l u^j)(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \|u^l(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\|u^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\| + \sum_{j=1}^n \|u^l(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\|u^j(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|.
\end{aligned}
\tag{5.2.5}
$$

Hence,

$$
\begin{aligned}
&\sum_{k,l=1}^n \|[\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\| \lesssim \sum_{k,l=1}^n \|u^l(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\|u^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\| \\
&= \left(\sum_{k=1}^n \|u^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|\right)^2 = \|\boldsymbol{u}|A^s_{p,q}(\mathbb{R}^n)_n\|^2.
\end{aligned}
$$

Now we can proceed similar to Lemma 3.3.2 and obtain

$$
\begin{aligned}
\int_{\mathcal{K}} |\boldsymbol{H}(x,t)|\mathrm{d}(x,t) &\lesssim \int_0^R \int_0^t (t-\tau)^{-\frac{d}{2\alpha}}\|\boldsymbol{u}|A^s_{p,q}(\mathbb{R}^n)_n\|^2 \mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim R^{2-\frac{1}{\alpha v}-\frac{d}{2\alpha}-\frac{a}{\alpha}}\,\|\boldsymbol{u}|L_{2\alpha v}((0,T),\,\frac{a}{2\alpha},\,A^s_{p,q}(\mathbb{R}^n)_n)\|^2.
\end{aligned}
$$

Thus, $G^\alpha * [\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k$ exists in the sense of $D'(\mathbb{R}^n\times(0,T))$ for all $k,l=1,\dots,n$ and is locally integrable in $\mathbb{R}^{n+1}$. Then there exists also

$$
\partial_i(G^\alpha * [\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k) = (\partial_i G^\alpha) * [\mathbb{P}(\boldsymbol{u}\otimes\boldsymbol{u})^l]^k = G^\alpha * (\mathbb{P}\partial_i(\boldsymbol{u}\otimes\boldsymbol{u})^l)^k
$$

in the sense of $D'(\mathbb{R}^n\times(0,T))$ where we used that partial derivatives and Riesz transform can be interchanged. The rest of the proof coincides with that of Lemma 3.3.3 replacing u, u^2 and Du^2 by their vector-valued counterparts. $\square$

Remark 5.2.2. In particular we have seen that considerations for the generalized Navier-Stokes equations can be reduced to the respective scalar counterpart. Thus, corresponding assertions for the remaining cases $A^s_{p,q}(\mathbb{R}^n)$ with $\frac{n}{p}-1<s<\frac{n}{p}$ or $s=\frac{n}{p}$ and $F^0_{p,2}(\mathbb{R}^n)=L_p(\mathbb{R}^n)$, $2\le n<p<\infty$ follow similarly replacing the estimates of the scalar norm $\|u^k(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|$ by the respective estimates in Lemmas 4.2.2, 4.3.2 and 4.4.2.

Theorem 5.2.3. *Let $n\in\mathbb{N}$, $n\ge 2$, $\alpha\in\mathbb{N}$, $1<p<\infty$, $1\le q\le\infty$ and s such that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra.*

(i) *Let*

$$0 < \lambda < g \leq 1, \ \frac{2}{\alpha} < v \leq \infty, \ a = \alpha - \frac{1}{v} - \alpha\lambda \tag{5.2.6}$$

and let $\boldsymbol{u}_0 \in A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)_n$ *for the initial data. Then there exists a number* $T > 0$ *such that*

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u}(x,t) + \mathbb{P}\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})(x,t) = 0, \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{5.2.7}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \text{in } \mathbb{R}^n \tag{5.2.8}$$

has a unique mild solution

$$\boldsymbol{u} \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)_n) \cap C^\infty(\mathbb{R}^n \times (0,T))_n.$$

(ii) *If, in addition,* $q < \infty$ *and*

$$\frac{g}{2} \leq \lambda < g \leq 1 \ \text{ if } \ v < \infty \quad \text{ and } \quad \frac{g}{2} < \lambda < g \leq 1 \ \text{ if } \ v = \infty \tag{5.2.9}$$

then the above solution is strong, that means $u \in C([0,T), A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)_n)$.

(iii) *Under the additional assumption on the parameter* λ *imposed in Part **(ii)** the solution* $\boldsymbol{u}$ *obtained in Part **(i)** is locally stable and hence the problem* (5.0.2) *well-posed in the setting of this theorem if* $q < \infty$.

Proof. **Step 1.** We assume $v < \infty$. Otherwise one has to modify appropriately. The idea is to show that the above theorem can be reduced to its scalar version, i.e. to Theorem 3.3.5. Using Proposition 3.2.3 with $d = \alpha$ and $s - d$ in place of s we obtain

$$\begin{aligned}
&\|T_{\boldsymbol{u}_0}\boldsymbol{u}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\| = \sum_{k=1}^n \|T_{u_0^k} u_k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\leq \sum_{k=1}^n t^{-\frac{\alpha-\alpha g}{2\alpha}} \|u_0^k|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&- \sum_{k=1}^n t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{a}{\alpha}} \left(\int_0^t \tau^{av}\|(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^k(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}.
\end{aligned} \tag{5.2.10}$$

We estimate the norm in (5.2.10). Hereby we use that the Riesz transform R_k with $R_k : A^s_{p,q}(\mathbb{R}^n) \to A^s_{p,q}(\mathbb{R}^n)$ is bounded under the above assumptions, cf. therefore Proposition 2.4.2.

$$\begin{aligned}
&\|(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^k(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \\
&\leq \|\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})^k(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| + \|R_k \sum_{j=1}^n R_j \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})^j(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \|\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})^k(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| + \sum_{j=1}^n \|\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})^j(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|.
\end{aligned}$$

Now we use the embedding $A^{s-1}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{s-\alpha}_{p,q}(\mathbb{R}^n)$, $\partial_j : A^s_{p,q}(\mathbb{R}^n) \to A^{s-1}_{p,q}(\mathbb{R}^n)$ is linear and bounded and the assumption that $A^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra. Then we obtain

$$\begin{aligned}
&\|(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^k(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \|\sum_{i=1}^n \partial_i(u^iu^k)(\cdot,t)|A^{s-1}_{p,q}(\mathbb{R}^n)\| + \sum_{j=1}^n \|\sum_{i=1}^n \partial_i(u^iu^j)(\cdot,t)|A^{s-1}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \sum_{i=1}^n \|(u^iu^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| + \sum_{i,j=1}^n \|(u^iu^j)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \sum_{i=1}^n \|u^i(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\|u^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| + \sum_{i,j=1}^n \|u^i(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\|u^j(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|.
\end{aligned} \tag{5.2.11}$$

Inserting (5.2.11) in (5.2.10) yields

$$\begin{aligned}
&\|T_{\boldsymbol{u}_0}\boldsymbol{u}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\| \lesssim t^{-\frac{1-g}{2}} \sum_{k=1}^n \|u_0^k|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&- t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{a}{\alpha}} \left(\int_0^t \tau^{av}\left(\sum_{i,k=1}^n \|u^i(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\|u^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{\alpha v} \mathrm{d}\tau\right)^{1/\alpha v} \\
&\lesssim t^{-\frac{1-g}{2}} \sum_{k=1}^n \|u_0^k|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)\| \\
&- t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{a}{\alpha}} \left(\int_0^t \tau^{av}\left(\sum_{k=1}^n \|u^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{2\alpha v} \mathrm{d}\tau\right)^{1/\alpha v} \\
&\lesssim t^{-\frac{1-g}{2}} \|\boldsymbol{u}_0|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)_n\| - t^{\frac{1}{2}-\frac{1}{\alpha v}-\frac{a}{\alpha}} \|\boldsymbol{u}|L_{2\alpha v}((0,T),\, \frac{a}{2\alpha},\, A^s_{p,q}(\mathbb{R}^n)_n)\|.
\end{aligned} \tag{5.2.12}$$

Let $\boldsymbol{u} \in U_T$ where U_T denotes now the unit ball in $L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A^s_{p,q}(\mathbb{R}^n)_n)$. We proceed as in Step 1 of Theorem 3.3.5 and obtain that $T_{\boldsymbol{u}_0} : U_T \to U_T$ is a self mapping for an appropriately small chosen $T > 0$.
As for the contraction assume $\boldsymbol{u}, \boldsymbol{v} \in U_T$. Then a similar calculation leads to

$$\begin{aligned}
&\|T_{\boldsymbol{u}_0}\boldsymbol{u}(\cdot,t) - T_{\boldsymbol{u}_0}\boldsymbol{v}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\| = \sum_{k=1}^n \|T_{u_0^k}u^k(\cdot,t) - T_{u_0^k}v^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \sum_{k=1}^n t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}} \times \\
&\times \left(\int_0^t \tau^{av}\|((\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^k - (\mathbb{P}\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})))^k(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}.
\end{aligned} \tag{5.2.13}$$

We estimate the norm in (5.2.13), which we denote by (I), using the same ideas as in (5.2.10) - (5.2.12).

$$
\begin{aligned}
(I) &\lesssim \|(\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})^k - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})^k)(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \\
&\quad + \|R_k \sum_{j=1}^n R_j(\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})^j - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})^j)(\cdot,t)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \|\sum_{i=1}^n \partial_i(u^iu^k - v^iv^k)(\cdot,t)|A^{s-1}_{p,q}(\mathbb{R}^n)\| + \sum_{j=1}^n \|\sum_{i=1}^n \partial_i(u^iu^j - v^iv^j)(\cdot,t)|A^{s-1}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim \sum_{i=1}^n \|(u^iu^k - v^iv^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| + \sum_{i,j=1}^n \|(u^iu^j - v^iv^j)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|. \qquad (5.2.14)
\end{aligned}
$$

Inserting this in (5.2.13) yields

$$
\begin{aligned}
&\|T_{\boldsymbol{u}_0}\boldsymbol{u}(\cdot,t) - T_{\boldsymbol{u}_0}\boldsymbol{v}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\| \le c\,t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}} \\
&\times \left(\int_0^t \tau^{av}\left(\sum_{i,k=1}^n \|(u^iu^k - v^iv^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{\alpha v} \mathrm{d}\tau\right)^{1/\alpha v}. \qquad (5.2.15)
\end{aligned}
$$

Estimating the norm in (5.2.15) separately using the multiplication properties of $A^s_{p,q}(\mathbb{R}^n)$ leads to

$$
\begin{aligned}
&\|(u^iu^k - v^iv^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \lesssim (\|u^i(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\|(u^k - v^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| \\
&+ \|v^k(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\|(u^i - v^i)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|). \qquad (5.2.16)
\end{aligned}
$$

We insert this in (5.2.15) and use Minkowski's inequality. Hence,

$$
\begin{aligned}
&\|T_{\boldsymbol{u}_0}\boldsymbol{u}(\cdot,t) - T_{\boldsymbol{u}_0}\boldsymbol{v}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\| \le c\,t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}} \\
&\times \left(\int_0^t \tau^{av}\left(\sum_{i=1}^n \|u^i(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{\alpha v}\left(\sum_{k=1}^n \|(u^k - v^k)(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\|\right)^{\alpha v} \mathrm{d}\tau\right)^{1/\alpha v} \\
&\le c\,t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}}\left(\int_0^t \tau^{av}\|\boldsymbol{u}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\|^{\alpha v}\|(\boldsymbol{u}-\boldsymbol{v})(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\|^{\alpha v}\mathrm{d}\tau\right)^{1/\alpha v}.
\end{aligned}
$$

Now we apply Hölder's inequality and obtain

$$
\begin{aligned}
&\|T_{\boldsymbol{u}_0}\boldsymbol{u}(\cdot,t) - T_{\boldsymbol{u}_0}\boldsymbol{v}(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)_n\| \\
&\le c\,t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}}\|\boldsymbol{u}|L_{2\alpha v}((0,T),\,\frac{a}{2\alpha},\,A^s_{p,q}(\mathbb{R}^n)_n)\|\|\boldsymbol{u}-\boldsymbol{v}|L_{2\alpha v}((0,T),\,\frac{a}{2\alpha},\,A^s_{p,q}(\mathbb{R}^n)_n)\| \\
&\le c\,t^{\frac{1}{2}-\frac{a}{\alpha}-\frac{1}{\alpha v}}\|\boldsymbol{u}-\boldsymbol{v}|L_{2\alpha v}((0,T),\,\frac{a}{2\alpha},\,A^s_{p,q}(\mathbb{R}^n)_n)\| \qquad (5.2.17)
\end{aligned}
$$

since $\boldsymbol{u} \in U_T$. Multiplication with $t^{\frac{a}{2\alpha}}$, raising to the power of $2\alpha v$ and integration over $t \in (0,T)$ shows that $T_{\boldsymbol{u}_0}$ is a contraction in U_T with an appropriately small chosen $T > 0$. Hence, $T_{\boldsymbol{u}_0}$ has a uniquely determined fixed point in U_T which is also a solution of (5.0.2).

Step 2. To extend the assertion of Step 1 to the whole space one has to follow the ideas of Step 2 - Step 4 in Theorem 3.3.5 iterating (5.2.10) - (5.2.12) and replacing the scalar norms by their vector-valued counterpart.

Step 3. To show that the solution $\boldsymbol{u}$, obtained in the previous steps, is strong consider

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)_n\| = \sum_{k=1}^n \|u^k(\cdot,t) - u_0^k|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\|$$
$$\lesssim \sum_{k=1}^n \left(\|W_t^\alpha u_0^k - u_0^k|A^{s-\alpha+\alpha g}_{p,q}(\mathbb{R}^n)\| - \int_0^t (t-\tau)^{\frac{g}{2}} \|(\mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}))^k(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau \right).$$

The first summand can be estimated component by component according to (3.3.29) - (3.3.31). Concerning the second summand we proceed analogously to (5.2.11). Thus, the situation is similar to (3.3.28) and leads to the same restrictions for the parameter λ.

Step 4. We show that the solution $\boldsymbol{u}$ is also stable in the sense of (3.4.2), (3.4.1). Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ be solutions with corresponding initial data $\boldsymbol{u}_0^1$, $\boldsymbol{u}_0^2$ in the respective time interval. Then we have

$$\|\boldsymbol{u}_1(\cdot,t) - \boldsymbol{u}_2(\cdot,t)|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)_n\| = \sum_{k=1}^n \|u_1^k(\cdot,t) - u_2^k(\cdot,t)|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)\|$$
$$\leq \sum_{k=1}^n \|W_t^\alpha(u_{0_1}^k - u_{0_2}^k)|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)\|$$
$$- \sum_{k=1}^n \int_0^t \|W_{t-\tau}^\alpha((\mathbb{P}\operatorname{div}(\boldsymbol{u}_1\otimes\boldsymbol{u}_1))^k - (\mathbb{P}\operatorname{div}(\boldsymbol{u}_2\otimes\boldsymbol{u}_2))^k)(\cdot,\tau)|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau$$
$$\leq \sum_{k=1}^n \|u_{0_1}^k - u_{0_2}^k|A^{s-\alpha-\alpha g}_{p,q}(\mathbb{R}^n)\|$$
$$- \sum_{k=1}^n \int_0^t (t-\tau)^{\frac{g}{2}} \|((\mathbb{P}\operatorname{div}(\boldsymbol{u}_1\otimes\boldsymbol{u}_1))^k - (\mathbb{P}\operatorname{div}(\boldsymbol{u}_2\otimes\boldsymbol{u}_2))^k)(\cdot,\tau)|A^{s-\alpha}_{p,q}(\mathbb{R}^n)\|\mathrm{d}\tau.$$

But then one is in the same situation as in Step 1 and can follow the arguments starting from the norm estimate in (5.2.13) with $\boldsymbol{u}_1$ in place of $\boldsymbol{u}$ and $\boldsymbol{u}_2$ in place of $\boldsymbol{v}$. The rest of the proof coincides exactly with that of Theorem 3.4.1 now with the vector-valued norm counterparts. □

We formulate the corresponding results for (5.0.2) underlying function spaces $A^s_{p,q}(\mathbb{R}^n)$ with $\frac{n}{p} - 1 < s < \frac{n}{p}$.

Theorem 5.2.4. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $s > 0$. Let $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$ and $\infty > r > n$ ($q \leq r$ for B-spaces).*

(i) *Let*

$$\frac{2n}{r} < \lambda < g \leq 1 + \frac{n}{r}, \quad 0 \leq \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{r}\right), \quad a = \alpha\left(1 + \frac{n}{r}\right) - \frac{1}{v} - \alpha\lambda \tag{5.2.18}$$

and let $u_0 \in A_{p,q}^{s-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)_n$ for the initial data. Then there exists a number $T > 0$ such that

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u}(x,t) + \mathbb{P}\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})(x,t) = 0, \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{5.2.19}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \text{in } \mathbb{R}^n \tag{5.2.20}$$

has a unique mild solution

$$\boldsymbol{u} \in L_{2\alpha v}((0,T), \frac{a}{2\alpha}, A_{p,q}^s(\mathbb{R}^n)_n) \cap C^\infty(\mathbb{R}^n \times (0,T))_n.$$

(ii) *If, in addition, $q < \infty$ for the B-spaces and*

$$\frac{g}{2} + \frac{n}{r} \leq \lambda < g \leq 1 + \frac{n}{r} \text{ if } v < \infty \text{ and } \frac{g}{2} + \frac{n}{r} < \lambda < g \leq 1 + \frac{n}{r} \text{ if } v = \infty \tag{5.2.21}$$

then the above solution is strong, that means $u \in C([0,T), A_{p,q}^{s-\alpha+\alpha g}(\mathbb{R}^n)_n)$.

(iii) *Under the additional assumption on the parameter λ imposed in Part **(ii)** the solution $\boldsymbol{u}$ obtained in Part **(i)** is locally stable and hence the problem* (5.0.2) *well-posed in the setting of this theorem if $q < \infty$.*

Proof. The proof is the same as that of Theorem 5.2.3 based on Proposition 4.2.1 instead of Proposition 3.2.3. □

For the limiting case $s = \frac{n}{p} > 0$ the results are as follows.

Theorem 5.2.5. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{N}$, $1 < p < \infty$, $1 \leq q \leq \infty$ ($1 < q$ for B-spaces) and* $\max(n,p) < r < \infty$.

(i) *Let*

$$\frac{2n}{r} < \lambda < g \leq 1 + \frac{n}{r}, \quad 0 \leq \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{r}\right), \quad a = \alpha\left(1 + \frac{n}{r}\right) - \frac{1}{v} - \alpha\lambda \tag{5.2.22}$$

and let $u_0 \in A_{p,q}^{\frac{n}{p}-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)_n$ *for the initial data. Then there exists a number* $T > 0$ *such that*

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u}(x,t) + \mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})(x,t) = 0, \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{5.2.23}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \text{in } \mathbb{R}^n \tag{5.2.24}$$

has a unique mild solution

$$\boldsymbol{u} \in L_{2\alpha v}((0,T),\ \frac{a}{2\alpha},\ A_{p,q}^{\frac{n}{p}}(\mathbb{R}^n)_n) \cap C^\infty(\mathbb{R}^n \times (0,T))_n.$$

(ii) *If, in addition,* $q < \infty$ *for the* B*-spaces and*

$$\frac{g}{2} + \frac{n}{r} \le \lambda < g \le 1 + \frac{n}{r} \text{ if } v < \infty \text{ and } \frac{g}{2} + \frac{n}{r} < \lambda < g \le 1 + \frac{n}{r} \text{ if } v = \infty \tag{5.2.25}$$

then the above solution is strong, that means $u \in C([0,T), A_{p,q}^{\frac{n}{p}-\alpha\left(1+\frac{n}{r}\right)+\alpha g}(\mathbb{R}^n)_n)$.

(iii) *Under the additional assumption on the parameter* λ *imposed in Part* ***(ii)*** *the solution* $\boldsymbol{u}$ *obtained in Part* ***(i)*** *is locally stable and hence the problem* (5.0.2) *well-posed in the setting of this theorem if* $q < \infty$.

Proof. The proof is the same as that of Theorem 5.2.3 based on Proposition 4.3.1 instead of Proposition 3.2.3. □

We conclude the chapter with the corresponding result in the $L_p(\mathbb{R}^n)$ case, $1 < p < \infty$.

Theorem 5.2.6. *Let* $\alpha \in \mathbb{N}$ *and* $n \in \mathbb{N}$ *with* $2 \le n < p < \infty$.

(i) *Let*

$$\frac{2n}{p} < \lambda < g \le 1 + \frac{n}{p}, \quad 0 \le \frac{1}{v} < \frac{\alpha}{2}\left(1 - \frac{n}{p}\right), \quad a = \alpha\left(1 + \frac{n}{p}\right) - \frac{1}{v} - \alpha\lambda \tag{5.2.26}$$

and let $u_0 \in F_{p,2}^{-\alpha\left(1+\frac{n}{p}\right)+\alpha g}(\mathbb{R}^n)_n$ *for the initial data. Then there exists a number* $T > 0$ *such that*

$$(\partial_t + (-\Delta_x)^\alpha)\boldsymbol{u}(x,t) + \mathbb{P}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})(x,t) = 0, \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{5.2.27}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \text{in } \mathbb{R}^n \tag{5.2.28}$$

has a unique mild solution

$$\boldsymbol{u} \in L_{2\alpha v}((0,T),\ \frac{a}{2\alpha},\ L_p(\mathbb{R}^n)_n) \cap C^\infty(\mathbb{R}^n \times (0,T))_n.$$

(ii) *If, in addition,*

$$\frac{g}{2} + \frac{n}{p} \leq \lambda < g \leq 1 + \frac{n}{p} \ \textit{if } v < \infty \ \textit{and} \ \frac{g}{2} + \frac{n}{p} < \lambda < g \leq 1 + \frac{n}{p} \ \textit{if } v = \infty \quad (5.2.29)$$

then the above solution is strong, that means $u \in C([0,T), F_{p,2}^{-\alpha\left(1+\frac{n}{p}\right)+\alpha g}(\mathbb{R}^n)_n)$.

(iii) *Under the additional assumption on the parameter* λ *imposed in Part **(ii)** the solution* $\boldsymbol{u}$ *obtained in Part **(i)** is locally stable and hence the problem* (5.0.2) *well-posed in the setting of this theorem.*

Proof. The proof is the same as that of Theorem 5.2.3 based on Proposition 4.4.1 instead of Proposition 3.2.3. □

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Jena, den 30. Januar 2017 Franka Baaske